Properties of characters of π-separable groups

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Nonvanishing elements

Characterizing nonvanishing elements

A known result of Burnside asserts that every nonlinear irreducible character vanishes on some conjugacy class. However, it is false that every conjugacy class different from $\{1\}$ has an irreducible character which vanishes on it.

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Suppose that a group G has a normal Sylow p-subgroup P. Then all the elements of $\mathbf{Z}(P)$ are nonvanishing in G.

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Theorem (Isaacs, Navarro, Wolf, 1999)

Let G be a finite supersolvable group and let $\mathbf{F}(G)$ be its Fitting subgroup, then all its nonvanishing elements lie in $\mathbf{Z}(\mathbf{F}(G))$.

Nonvanishing elements in solvable groups

Theorem (Isaacs, Navarro, Wolf, 1999)

Let G be a finite solvable group and let $x \in G$ be a nonvanishing element. Then the image of x in $G/\mathbf{F}(G)$ under the canonical epimorphism has 2-power order.

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The key to get this result are three facts:

- the Fitting subgroup is equal to the intersection of the centralizers of each chief factor of the group;
- if $N \lhd G$, a nonvanishing element fixes some member in every orbit of the action of G over $\mathrm{Irr}(N)$;
- if G is a nilpotent subgroup, G' is cyclic, and G acts on a finite vector space V faithfully and irreducibly, then, for some $v \in V$, $|C_G(v)| \leq 2$.

Then the result is obtained by induction, by studying the action of a nonvanishing element on a chief factor.

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If x is a π-nonvanishing element of G of odd order, in general it does not lie in $\mathbf{F}_\pi(G)$.

Example

Let $G = \mathrm{GL}(2,3) \ltimes {\mathbb{Z}_3}^2$, then $\mathbf{F}(G) = \mathbf{F}_3(G) = {\mathbb{Z}_3}^2$ and there is a conjugacy class of nonvanishing elements of size 8, however there are also some other $\{3\}$-nonvanishing elements of order 3, which, however, are not contained in $\mathbf{F}(G)$.

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Proposition

If G is a π-supersolvable group and $x \in G$ is π-nonvanishing, then x centralizes every chief factor of G which is also a π-groups.

We may consider a larger set

This suggests that we may study the elements which are both π-nonvanishing and π'-nonvanishing.

Proposition

Let G be a finite solvable group, let $x \in G$ and suppose that no character in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ vanishes on x. Then the image of x in $G/\mathbf{F}(G)$ under the canonical epimorphism has 2-power order.
If G is supersolvable, then x is contained in $\mathbf{Z}(\mathbf{F}(G))$.

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If G is supersolvable, then x is contained in $\mathbf{Z}(\mathbf{F}(G))$.

The reason why the proof of the results of Isaacs, Navarro and Wolf can be used also in this case, with only small changes, is the following simple fact:

Lemma

Let G be π-separable, let $N \trianglelefteq G$ and let $\psi \in \mathrm{B}_\pi(N)$, then there exists a character $\chi \in \mathrm{Irr}(G|\psi)$ which belongs to $\mathrm{B}_\pi(G)$.

Primes and normal Sylow subgroups

Ito-Michler Theorem

Theorem (Ito-Michler)

Let G be a finite group and let p be a prime. Let P be a Sylow p-subgroup of G. Then, P is normal in G and abelian if and only if $p \nmid \chi(1)$ for any $\chi \in \mathrm{Irr}(G)$.

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Can we relate similar properties to a prime dividing the degree of the irreducible characters? When is P normal and not abelian? When abelian but not normal? And, on the other hand, what happens if the prime p does not divide the degree of only a subfamily of irreducible characters?

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Theorem (Michler, 1986)

Let G be a finite group and p a prime. Let P be a Sylow p-subgroup of G. Then, $P \lhd G$ if and only if $p \nmid \varphi(1)$ for any $\varphi \in \mathrm{IBr}_p(G)$.

Are there analogue results for the ordinary characters?

A connection between the apparently unrelated problems of finding variations of the Ito-Michler theorem and of characterising the nonvanishing elements is given by two important theorems.

Theorem (Navarro, 1998)

Let $\chi \in \mathrm{Irr}(G)$ and suppose $p \nmid \chi(1)$. Suppose $x \in G$ is a p-element, i.e., it has p-power order. Then, $\chi(x) \neq 0$.

Character degrees and nonvanishing elements

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Theorem (Dolfi, Pacifici, Sanus, Spiga, 2009)

Let G be a finite group and p a prime number. If all the p-elements of G are nonvanishing, then G has a normal Sylow p-subgroup.

These results have been the key for further developments in the search of Ito-Michler type results.

Ito-Michler for subsets of the irreducible characters

Using the same technique as in the article of Dolci, Pacifici, Sanus and Spiga, it has been given two conditions that guarantee that the group has a normal Sylow p-subgroup.

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The first one is a necessary and sufficient condition and regards the irreducible constituents of the character induced from the regular character of a Sylow p-subgroup.

Theorem (Malle, Navarro, 2012)

Let G be a finite group, let p be a prime and let $P \in \mathrm{Syl}_p(G)$. Let $\mathrm{Irr}(1_P{}^G)$ be the set of irreducible constituents of the character $(1_P)^G$. Then the following conditions are equivalent:

a) *p does not divide $\chi(1)$ for all the characters $\chi \in \mathrm{Irr}(1_P{}^G)$;*
b) *$\chi(x) \neq 0$ for all the characters $\chi \in \mathrm{Irr}(1_P{}^G)$ and all $x \in P$;*
c) *$P \lhd G$.*

p-rational characters and normal Sylow p-subgroups

The second condition that guarantees the existence of a normal Sylow p-subgroup regards the p-rational characters. Remember that a character χ is said to be p-rational if its field of values is contained in $\mathbb{Q}_{p'}$.

Theorem (Navarro, Tiep, 2012)

Let G be a finite group, let p be a prime and let $P \in \mathrm{Syl}_p(G)$. If no p-rational character vanishes on some p-elements of G, then $P \lhd G$.
In particular, if $p \nmid \chi(1)$ for all p-rational $\chi \in \mathrm{Irr}(G)$, then $P \lhd G$.

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Observation

The previous theorems involve the irreducible constituents of $(1_P)^G$ and the p-rational characters. This is interesting, because it is known that the B_π-characters are π'-rational. Moreover, if $|G|$ is odd, then the B_π-characters are exactly the π'-rational irreducible characters, and they are also the irreducible constituents of $(1_H)^G$ of odd multiplicity, where H is an Hall π'-subgroup (Isaacs, 1991).

Results of Ito-Michler type for B_π-characters

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Proposition

Let G be a finite π-separable group, then G has a normal Hall π'-subgroup if and only if the degree of every character in $\mathrm{B}_\pi(G)$ is divided only by primes in π.

Moreover, one can notice that, if the Hall π'-subgroup is normal, then each of its irreducible characters is a Fong character. Thus, the normal Hall subgroup is also abelian if and only if also the degree of every character in $\mathrm{B}_{\pi'}(G)$ is divided only by primes in π.

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The connection with the abelianity is actually more general.

Proposition

Let G be a finite π-separable group and let p be any prime. Let P be a Sylow p-subgroup of G. Then, P is normal in G and abelian if and only if $p \nmid \chi(1)$ for every $\chi \in \mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$.

Self-normalizing subgroup

Once one knows when a Sylow subgroup is normal, it could be natural to ask when, on the other hand, is self-normalizing.

Theorem

Let G be a finite group and P a Sylow p-subgroup of G, with p odd. Then, $N_G(P) = P$ if and only if, either:

- *p divides the degree of every non-trivial p-rational irreducible character of G (Navarro, Tiep, Turull, 2007);*
- *p divides the degree of every non-trivial irreducible p-Brauer character (Navarro, Tiep, 2010).*

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- *p divides the degree of every non-trivial p-rational irreducible character of G (Navarro, Tiep, Turull, 2007);*
- *p divides the degree of every non-trivial irreducible p-Brauer character (Navarro, Tiep, 2010).*

Proposition

Let G be either a π-solvable or a π'-solvable group and let H be a Hall π-subgroup. Then, $N_G(H) = H$ if and only if the degree of every non-trivial character in $\mathrm{B}_{\pi'}(G)$ is divisible by some prime in π.

The McKay correspondence

The McKay conjecture

Conjecture (McKay, 1972)

Let G be a finite group, p a prime. Let $P \in \mathrm{Syl}_p(G)$ and $N = N_G(P)$. Then $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(N)|$.

The conjecture has been solved for p-solvable groups, reduced to simple groups (Isaacs, Malle, Navarro, 2007), and solved for any group if $p = 2$ (Malle, Spath, 2015).

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A related problem, however, is to find a *naturally defined* McKay correspondence. This means to find a uniquely defined function between the sets of $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(N)$ which determines the bijection of the conjecture.
In few cases a natural correspondence has been proved to exist:

- for p odd, when $N_G(P) = PC_G(P)$, for the characters in the principal block (Navarro, Tiep, Vallejo, 2014);
- for $p = 2$, when G is the symmetric group (Giannelli, 2015).

However, the problem is still open even for solvable groups.

B_π-characters associated with a primitive Fong character

Proposition (Isaacs, 1984)

Let G be π-separable with Hall π-subgroup H. Suppose $\alpha \in \mathrm{Irr}(H)$ is primitive. Then α is a Fong character for the group G and if β is another Fong character associated with the same $\chi \in \mathrm{B}_\pi(G)$, then α and β are $N_G(H)$-conjugate.

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Let $\mathcal{B}(G)$ be the set of irreducible B_π-characters of a group G which have an associated primitive Fong character. Let one call $\mathcal{B}_k(G)$ the characters in $\mathcal{B}(G)$ with a degree which π-part is equal to k.

Corollary

Let G be a π-separable group, H be a Hall π-subgroup and let $N = N_G(H)$. Then, $|\mathcal{B}_k(G)| = |\mathcal{B}_k(N)|$ for every k.

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Let G be a π-separable group, H be a Hall π-subgroup and let $N = N_G(H)$. Then, $|\mathcal{B}_k(G)| = |\mathcal{B}_k(N)|$ for every k.

Corollary

Let G be a p-solvable group and let P be a Sylow p-subgroup. Then, $N_G(P) = P$ if and only if p divides the degree of every irreducible character of G which does not belong to $\mathrm{B}_p(G)$.

Natural correspondence between B_π-characters

Let $\chi \in \mathrm{B}_\pi(G)$ and let $N = N_G(H)$. If ψ is an irreducible constituent of χ_N which lies over a Fong character, then $\psi(1)_\pi = \chi(1)_\pi$ and $[\chi_N, \psi] = 1$. We can call ψ an *upper-Fong character* associated with χ.
If $\chi \in \mathcal{B}(G)$, then there exists a unique upper-Fong character associated with χ.

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Corollary

Let G be p-solvable, then the character restriction realizes a natural McKay correspondence between $\mathrm{B}_p(G) \cap \mathrm{Irr}_{p'}(G)$ and $\mathrm{B}_p(N) \cap \mathrm{Irr}_{p'}(N)$.

Thank you!