Ergodic properties of the multiplication operator

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An operator $T \in \mathcal{L}(E)$, is called **power bounded** if $\{T^n\}_{n=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(E)$.

Given $T \in \mathcal{L}(E)$, the averages are $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, n \in \mathbb{N}$.
The operator T is said to be **(uniformly) mean ergodic** if $\{T_{[n]}\}_{n=1}^{\infty}$ converges in $\mathcal{L}_s(E)$ (resp. $\mathcal{L}_b(E)$).

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Klilou, Manhas, Oubbi, Singh 2000's
For holomorphic functions on $\mathbb{D}$: Bonet and Ricker 2009

Power Boundedness

T_φ is power bounded $\iff$ S_φ is power bounded $\iff$ $\|\varphi\|_\infty \leq 1$

Mean Ergodicity

Assume $\|\varphi\|_\infty \leq 1$.

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$$((M_\varphi)_{[n]}f)(x) = \begin{cases} \frac{f(x)\varphi(x)}{n}\frac{1-\varphi(x)^n}{1-\varphi(x)}, & \text{if } \varphi(x) \neq 1 \\ f(x), & \text{if } \varphi(x) = 1 \end{cases}$$

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$$h_f(x) = \begin{cases} 0, & \text{if } \varphi(x) \neq 1 \\ f(x), & \text{if } \varphi(x) = 1 \end{cases}$$

Mean Ergodicity in C_v and C_v^0

Theorem

$S_\varphi \in \mathcal{L}(C_v^0)$ is mean ergodic $\iff$ $\|\varphi\|_\infty \le 1$ and $\varphi^{-1}(1)$ is open.

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Theorem

The following assertions are equivalent:

1. $T_\varphi \in \mathcal{L}(C_v)$ *is mean ergodic,*
2. $T_\varphi \in \mathcal{L}(C_v)$ *is uniformly mean ergodic,*
3. $S_\varphi \in \mathcal{L}(C_v^0)$ *is uniformly mean ergodic,*
4. $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ *is open and*

$$\inf\{|1-\varphi(x)| \, : \, x \in X \setminus \varphi^{-1}(1)\} > 0.$$

Inductive Limits

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Nachbin Family

$$\overline{V} := \{\overline{v} : X \longrightarrow (0,\infty) \ : \overline{v} \text{ is upper semicontinuous and } \frac{\overline{v}}{v_k} \text{ is bounded on } X \, \forall k \in \mathbb{N}\}.$$

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Bierstedt, Bonet, Meise, Summers 1980's

Continuity

$$M_\varphi \in \mathcal{L}(VC) \iff M_\varphi \in \mathcal{L}(V_0C) \iff \forall k \exists \ell : \sup_{x \in X} \frac{v_\ell(x)}{v_k(x)} |\varphi(x)| < \infty$$

Power Boundedness

$$M_\varphi \in \mathcal{L}(VC) \text{ power bounded } \iff M_\varphi \in \mathcal{L}(V_0C) \text{ power bounded}$$

$$\iff \|\varphi\|_\infty \leq 1$$

Mean Ergodicity

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$M_\varphi \in \mathcal{L}(V_0C)$ is mean ergodic $\iff \|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.

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$M_\varphi \in \mathcal{L}(VC)$ is mean ergodic $\iff$ $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic.

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If V has property (S), then: $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic $\iff$ $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.

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Theorem

If V has property (S), then: $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic $\iff$ $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.

(S) $\iff$ $VC = V_0C$

Mean Ergodicity using $\overline{V}$

Theorem

$M_\varphi \in \mathcal{L}(V_0C)$ is uniformly mean ergodic $\iff$ $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open in X and for every $k \in \mathbb{N}$ and every $\overline{v} \in \overline{V}$,

$$\lim_{n\to\infty} \sup_{x \in Y} \frac{1}{n} \frac{\overline{v}(x)}{v_k(x)} |\varphi(x)| \frac{|1 - \varphi(x)^n|}{|1 - \varphi(x)|} = 0,$$

where $Y = X \setminus \varphi^{-1}(1)$.

Mean Ergodicity using (D)

Theorem

If V has property (D), then: $M_\varphi \in \mathcal{L}(VC)$ is uniformly mean ergodic $\iff$ $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ is open in X and for every $k \in \mathbb{N}$ and every $\overline{v} \in \overline{V}$,

$$\lim_{n\to\infty} \sup_{x\in Y} \frac{1}{n} \frac{\overline{v}(x)}{v_k(x)} |\varphi(x)| \frac{|1-\varphi(x)^n|}{|1-\varphi(x)|} = 0,$$

where $Y = X \setminus \varphi^{-1}(1)$.

(D) $\iff$ $VC = C\overline{V}$

Projective Limits

$A = (a_k)_k$ is an increasing family of continuous weights.

$$CA = \operatorname*{proj}_{n \in \mathbb{N}} C_{a_n}, \quad CA_0 = \operatorname*{proj}_{n \in \mathbb{N}} C^0_{a_n}$$

Continuity

$$M_\varphi \in \mathcal{L}(CA) \iff M_\varphi \in \mathcal{L}(CA_0) \iff \forall \ell \exists k : \sup_{x \in X} \frac{a_\ell(x)}{a_k(x)} |\varphi(x)| < \infty$$

Power Boundedness

$$M_\varphi \in \mathcal{L}(CA) \text{ power bounded } \iff M_\varphi \in \mathcal{L}(CA_0) \text{ power bounded}$$

$$\iff \|\varphi\|_\infty \leq 1$$

Mean Ergodicity

Theorem

$M_\varphi \in \mathcal{L}(CA_0)$ is mean ergodic $\iff$ $\|\varphi\|_\infty \leq 1$ and $\varphi^{-1}(1)$ is open.

Theorem

If X is even σ-compact, then the following assertions are equivalent:

1. $M_\varphi \in \mathcal{L}(CA)$ *is mean ergodic,*
2. $M_\varphi \in \mathcal{L}(CA)$ *is uniformly mean ergodic,*
3. $M_\varphi \in \mathcal{L}(CA_0)$ *is uniformly mean ergodic,*
4. $\|\varphi\|_\infty \leq 1$, $\varphi^{-1}(1)$ *is open in* X *and for every* $k \in \mathbb{N}$ *and every* $\overline{v} \in \overline{V}$,

$$\lim_{n\to\infty} \sup_{x\in Y} \frac{\overline{v}(x)a_k(x)}{n} |\varphi(x)| \frac{|1-\varphi(x)^n|}{|1-\varphi(x)|} = 0,$$

where $Y = X \setminus \varphi^{-1}(1)$.

Spectrum

E l.c.H.s.

The **resolvent set** $\rho(T, X)$ of $T \in \mathcal{L}(X)$ consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.
The **spectrum** of T is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$.
The **point spectrum** is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective.

Spectrum

Proposition

Let E be either C_v, C_v^0, VC, V_0C, CA or CA_0, for a weight v. Then

$$\sigma_{pt}(M_\varphi, E) \subset \{\varphi(x) \,:\, x \in X\} \subset \sigma(M_\varphi, E).$$

Spectrum

Proposition

Let E be either C_v, C_v^0, VC, V_0C, CA or CA_0, for a weight v. Then

$$\sigma_{pt}(M_\varphi, E) \subset \{\varphi(x) \,:\, x \in X\} \subset \sigma(M_\varphi, E).$$

Proposition

Let E be either C_v or C_v^0, for a weight v. Then

$$\sigma(M_\varphi, E) = \overline{\{\varphi(x) \,:\, x \in X\}}.$$

Spectrum

Proposition

Let E be either VC, V_0C, CA or CA_0. Then the following assertions are equivalent:

(1) $\mu \in \rho(M_\varphi, E)$,

(2) $M_\phi \in \mathcal{L}(X)$, *with* $\phi(\cdot) = \frac{1}{\varphi(\cdot)-\mu}$,

(3) *for each* k, *there exists* $l \geq k$ *such that*

$$\sup_{x\in X} \frac{v_l(x)}{v_k(x)} \cdot \frac{1}{|\varphi(x)-\mu|} = \sup_{x\in X} \frac{a_k(x)}{a_l(x)} \cdot \frac{1}{|\varphi(x)-\mu|} < \infty.$$

*-spectrum of Waelbroeck

E l.c.H.s.

$\rho^*(T,E)$ consists of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that each $\mu \in B(\lambda,\delta)$ belongs as well to $\rho(T,E)$ and such that the set $\{(\mu I - T)^{-1} : \mu \in B(\lambda,\delta)\}$ is equicontinuous in $\mathcal{L}(E)$.

$\sigma^*(T,E) := \mathbb{C} \setminus \rho^*(T,E)$.

Theorem

Let E be either VC, V_0C, CA or CA_0. Then

$$\sigma^*(M_\varphi, E) = \overline{\sigma(M_\varphi, E)}$$

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