On the extension of Whitney ultrajets II

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General notions

(∗) E will always denote a compact set in $\mathbb{R}^n$.

(∗) j_E^∞ is the jet mapping sending a smooth function to the infinite jet consisting of its partial derivatives of all orders (restricted to E).

(∗) Let $\omega : [0, +\infty) \to [0, +\infty)$ be a weight function (satisfying some basic growth properties), let $\mathcal{B}^{\{\omega\}}(\mathbb{R}^n)$ be the associated space of ultradifferentiable functions f on $\mathbb{R}^n$; i.e. the growth of $(\|f^{(\alpha)}\|_{L^\infty(\mathbb{R}^n)})_{\alpha\in\mathbb{N}^n}$ is regulated in terms of ω. (The letter $\mathcal{B}$ emphasizes that the bounds are global in $\mathbb{R}^n$.)

(∗) $\mathcal{B}^{\{\omega\}}(E)$ shall denote the space of jets on $E \subseteq \mathbb{R}^n$ (with a growth rate regulated by ω), so-called ultrajets.

Characterization result for preserving the class - Bonet/Braun/Meise/Taylor '91

Theorem

Let ω be a weight function (with some standard growth properties), then TFAE:

- (i) *For every compact $E \subseteq \mathbb{R}^n$ the jet mapping $j^\infty_E : \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \to \mathcal{B}^{\{\omega\}}(E)$ is surjective.*
- (ii) *ω is strong, i.e., $\int_1^\infty \frac{\omega(tu)}{u^2} du \le C\omega(t) + C$ for all $t > 0$ and some $C > 0$.*

Remark: This result holds true for the so-called Beurling case $\mathcal{B}^{(\omega)}$ as well (method there: using a reduction to the Roumieu case).

Known results I

Take ω such that $\int_1^\infty \frac{\omega(t)}{t^2}dt < +\infty$, let σ be another weight function.

Question: Under which conditions is j^∞_E defined on $\mathcal{B}^{\{\omega\}}(\mathbb{R}^n)$ surjective onto $\mathcal{B}^{\{\sigma\}}(E)$ for all compact sets $E \subseteq \mathbb{R}^n$?

(*i*) Bonet, Meise, and Taylor '92 - for the one-point set $E = \{0\}$

(*ii*) Langenbruch '94 - for compact convex E

The mapping $j^\infty_E : \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \to \mathcal{B}^{\{\sigma\}}(E)$ is surjective if and only if

$$\exists\, C > 0\ \forall\, t \geq 0 : \quad \int_1^\infty \frac{\omega(tu)}{u^2}du \leq C\sigma(t) + C. \tag{1}$$

(1) is the mixed strong non-quasianalyticity condition for weight functions.

Question: What can be said about general compact sets $E \subseteq \mathbb{R}^n$?

Known results II

First answer (Rainer/S. '17):

Theorem

Let ω be a non-quasianalytic concave weight function.

Let σ be a weight function satisfying $\sigma(t) = o(t)$ as $t \to \infty$ and $\mathcal{S} = \{S^x : x > 0\}$, the matrix associated with σ, has the good property.
Then the following conditions are equivalent:

(i) *For every compact $E \subseteq \mathbb{R}^n$ the jet mapping $j_E^\infty : \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \to \mathcal{B}^{\{\sigma\}}(E)$ is surjective.*

(ii) *There is $C > 0$ such that $\int_1^\infty \frac{\omega(tu)}{u^2} du \leq C\sigma(t) + C$ for all $t > 0$.*

Known results III

For the proof we have had to assume the following three additional conditions:

(*i*) ω is concave: Each strong weight function is equivalent to a concave one by Meise, and Taylor '88.

(*ii*) $\sigma(t) = o(t)$ as $t \to \infty$: Any strong weight function has this property.

(*iii*) (!!!) The weight matrix $\mathcal{S} = \{S^x : x > 0\}$ associated with σ satisfies the "good property":

$$\forall\, x > 0\; \exists\, y > 0\; \exists\, C \geq 1\; \forall\, 1 \leq j \leq k : \quad \frac{S_j^x}{jS_{j-1}^x} \leq C\frac{S_k^y}{kS_{k-1}^y}.$$

Using the weight matrix notation we have (Rainer, S. '14):

$$\mathcal{B}^{\{\sigma\}}(\mathbb{R}^n) = \varinjlim_{x>0} \mathcal{B}^{\{S^x\}}(\mathbb{R}^n).$$

Questions and main goal

Main goal: Get rid of the additional and undesired assumption that $\mathcal{S}$ has the good property - or see that this property is not "exotic", more precisely we have asked:

(∗) Is every concave weight function (equivalent to) a good one?

(∗) Is every strong weight function (equivalent to) a good one?

Note: Concavity and the good property are not invariant under equivalence, it will be enough to require these properties up to equivalence of weight functions, i.e. $\omega_1(t) = O(\omega_2(t))$, $\omega_2(t) = O(\omega_1(t))$ as $t \to +\infty$ and write $\omega_1 \sim \omega_2$.

Questions and main goal - some hope

Theorem

Let ω be a weight function and let $\mathcal{W}=\{W^x : x>0\}$ be the associated weight matrix, set $w^x_k := \frac{W^x_k}{k!}$, then TFAE:

(i) *ω is equivalent to its least concave majorant.*

(ii) $\forall\, x>0\ \exists\, y>0\ \exists\, D\ge 1\ \forall\, 1\le j\le k: \quad (w^x_j)^{1/j}\le D(w^y_k)^{1/k}$.

If $\mathcal{W}$ does have the strange growth property (of Roumieu type)

$$\forall\, x>0\ \exists\, y>0\ \exists\, C\ge 1\ \forall\, k\in\mathbb{N}_{\ge 1}: \quad \frac{W^x_k}{W^x_{k-1}}\le C(W^y_k)^{1/k},$$

then ω is good if and only if it is equivalent to its least concave majorant.

Some comments on the proof of the main theorem I

We have combined methods/techniques from BBMT '91 and Chaumat, and Chollet '94, more precisely we have

(∗) generalized special cut-off functions as constructed in BBMT '91 to a mixed setting,

(∗) combined the resulting partition of unity $\{\varphi_i\}_i$ subordinate to a collection of *Whitney cubes* $(Q_i)_i$ with centers $(x_i)_i$ with the technique of Ch./Ch. '94 - *mixed weight sequence setting* (based on an extension method of Dynkin '80),

(∗) that the extension of an ultrajet $F \in \mathcal{B}^{\{\sigma\}}$ is defined as a linear combination

$$\sum_i \varphi_i T_{\hat{x}_i}^{p(x_i)} F$$

of Taylor polynomials, where the degree $p(x_i)$ depends on the distance of x_i to E and $\hat{x}_i \in E$ realizes this distance.

Some comments on the proof of the main theorem II

(∗) The dependence of p is given through a counting function corresponding to the sequences in $\mathcal{S} = \{S^x :> 0\}$, the good property is only needed for this step.

(∗) We have worked with two counting functions, generalizing the technique of Ch./Ch. '94.

(∗) This has become necessary in order to exploit the fact that we can work with the matrix $\mathcal{S}$ associated with σ. Because:

(∗) In general some/each S^x will NOT have the property that $j \mapsto \frac{S^x_j}{jS^x_{j-1}}$ is increasing, i.e. S^x will be in general NOT strongly log-convex (as assumed in Ch./Ch. '94).

(∗) Finally we can not expect that some/each S^x will have condition (mg) - but this failure has been compensated by mixed (mg)-conditions which are satisfied automatically in this framework.

Weight functions

By a weight function we mean a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ and $\lim_{t\to\infty} \omega(t) = \infty$ that satisfies

$$\omega(2t) = O(\omega(t)) \quad \text{as } t \to \infty, \tag{2}$$

$$\omega(t) = O(t) \quad \text{as } t \to \infty, \tag{3}$$

$$\log t = o(\omega(t)) \quad \text{as } t \to \infty, \tag{4}$$

$$\varphi_\omega(t) := \omega(e^t) \text{ is convex.} \tag{5}$$

A weight function is called non-quasianalytic if

$$\int_0^\infty \frac{\omega(t)}{1+t^2}\,dt < \infty. \tag{6}$$

For each ω there is an equivalent $\tilde{\omega}$ such that $\omega(t) = \tilde{\omega}(t)$ for large $t > 0$ and $\tilde{\omega}|_{[0,1]} = 0$ (normalization).

The *Young conjugate* φ_ω^* of φ_ω is defined by

$$\varphi_\omega^*(t) := \sup_{s \geq 0}\{st - \varphi_\omega(s)\}, \quad t \geq 0.$$

Assuming $\omega|_{[0,1]} = 0$, we have that φ_ω^* is a convex increasing function satisfying $\varphi_\omega^*(0) = 0$, $t/\varphi_\omega^*(t) \to 0$ as $t \to \infty$, and $\varphi_\omega^{**} = \varphi_\omega$.

With each normalized ω we associate a weight matrix $\mathcal{W} = \{W^x : x > 0\}$ by

$$W_k^x := \exp\left(\frac{1}{x}\varphi_\omega^*(xk)\right).$$

The space $\mathcal{B}^{\{\omega\}}(\mathbb{R}^n)$

Let ω be a weight function and $l > 0$. Consider the Banach space

$$\mathcal{B}^\omega_l(\mathbb{R}^n) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n) : \|f\|^\omega_l < \infty\},$$

where

$$\|f\|^\omega_l := \sup_{x\in\mathbb{R}^n, \alpha\in\mathbb{N}^n} \frac{|f^{(\alpha)}(x)|}{\exp(\frac{1}{l}\varphi^*_\omega(l|\alpha|))},$$

and the inductive limit

$$\mathcal{B}^{\{\omega\}}(\mathbb{R}^n) := \varinjlim_{l\in\mathbb{N}_{>0}} \mathcal{B}^\omega_l(\mathbb{R}^n).$$

For weight functions ω and σ we have $\mathcal{B}^{\{\omega\}} \subseteq \mathcal{B}^{\{\sigma\}}$ if and only if $\sigma(t) = O(\omega(t))$ as $t \to \infty$.

Weight sequences

Let $\mu = (\mu_k)_k$ be a positive increasing sequence with $1 = \mu_0$, the sequences $M = (M_k)_k$ and $m = (m_k)_k$ are defined by

$$\mu_0 \mu_1 \mu_2 \cdots \mu_k = M_k = k! \cdot m_k.$$

We call M a *weight sequence*, if $M_k^{1/k} \to \infty$ as $k \to \infty$.

(i) M is log-convex,

(ii) $(M_k)^{1/k} \leq \mu_k$ follows,

(iii) but M is NOT assumed necessarily to be strongly log-convex, i.e. m is log-convex resp. $j \mapsto \frac{M_j}{jM_{j-1}} = \frac{\mu_j}{j}$ is increasing.

Each W^x is a weight sequence but NOT strongly log-convex in general.

The space $\mathcal{B}^{\{M\}}(\mathbb{R}^n)$

Let $M = (M_k)$ be a weight sequence and $\varrho > 0$. We consider the Banach space

$$\mathcal{B}^M_\varrho(\mathbb{R}^n) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n) : \|f\|^M_\varrho < \infty\},$$

where

$$\|f\|^M_\varrho := \sup_{x\in\mathbb{R}^n,\alpha\in\mathbb{N}^n} \frac{|f^{(\alpha)}(x)|}{\varrho^{|\alpha|} M_{|\alpha|}},$$

and the inductive limit

$$\mathcal{B}^{\{M\}}(\mathbb{R}^n) := \varinjlim_{\varrho\in\mathbb{N}_{>0}} \mathcal{B}^M_\varrho(\mathbb{R}^n).$$

Traditionally, $\mathcal{B}^{\{M\}}(\mathbb{R}^n)$ is called a *Denjoy–Carleman class* (of Roumieu type).

Beurling-type classes

If we consider the projective limits

$$\mathcal{B}^{(\omega)}(\mathbb{R}^n) := \varprojlim_{l\in\mathbb{N}_{>0}} \mathcal{B}^{\omega}_{1/l}(\mathbb{R}^n),$$

resp.

$$\mathcal{B}^{(M)}(\mathbb{R}^n) := \varprojlim_{\varrho\in\mathbb{N}_{>0}} \mathcal{B}^{M}_{1/\varrho}(\mathbb{R}^n),$$

then we obtain the ultradifferentiable classes of *Beurling-type.*

The connection between $\mathcal{B}^{\{\omega\}}(\mathbb{R}^n)$ and $\mathcal{B}^{\{M\}}(\mathbb{R}^n)$

Theorem

Let ω be a weight function, then, as locally convex spaces,

$$\mathcal{B}^{\{\omega\}}(\mathbb{R}^n) = \varinjlim_{x>0} \mathcal{B}^{\{W^x\}}(\mathbb{R}^n) = \varinjlim_{x>0} \varinjlim_{\varrho>0} \mathcal{B}^{W^x}_{\varrho}(\mathbb{R}^n).$$

$$\mathcal{B}^{(\omega)}(\mathbb{R}^n) = \varprojlim_{x>0} \mathcal{B}^{(W^x)}(\mathbb{R}^n) = \varprojlim_{x>0} \varprojlim_{\varrho>0} \mathcal{B}^{W^x}_{\varrho}(\mathbb{R}^n).$$

More generally, a weight matrix $\mathcal{M} := \{M^x : x > 0\}$ is a family of weight sequences such that $M^x \le M^y$ for all $x \le y$. We put

$$\mathcal{B}^{\{\mathcal{M}\}}(\mathbb{R}^n) := \varinjlim_{x>0} \mathcal{B}^{\{M^x\}}(\mathbb{R}^n)$$

and

$$\mathcal{B}^{(\mathcal{M})}(\mathbb{R}^n) := \varprojlim_{x>0} \mathcal{B}^{(M^x)}(\mathbb{R}^n).$$

Whitney ultrajets I

We denote by $\mathcal{J}^\infty(E)$ the vector space of all jets $F = (F^\alpha)_{\alpha\in\mathbb{N}^n} \in \mathcal{C}^0(E,\mathbb{R})^{\mathbb{N}^n}$ on E.

For $a \in E$ and $p \in \mathbb{N}$ we associate the Taylor polynomial

$$T_a^p : \mathcal{J}^\infty(E) \to \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}), \quad F \mapsto T_a^p F(x) := \sum_{|\alpha|\le p} \frac{(x-a)^\alpha}{\alpha!} F^\alpha(a),$$

and the remainder $R_a^p F = ((R_a^p F)^\alpha)_{|\alpha|\le p}$ with

$$(R_a^p F)^\alpha(x) := F^\alpha(x) - \sum_{|\beta|\le p-|\alpha|} \frac{(x-a)^\beta}{\beta!} F^{\alpha+\beta}(a), \quad a, x \in E.$$

Whitney ultrajets II

Let M be a weight sequence. For fixed $\varrho > 0$ we denote by $\mathcal{B}^M_\varrho(E)$ the set of all jets F such that there exists $C > 0$ with

$$|F^\alpha(a)| \le C\varrho^{|\alpha|}M_{|\alpha|}, \quad \alpha \in \mathbb{N}^n, \ \ a \in E,$$

$$|(R^p_a F)^\alpha(b)| \le C\varrho^{p+1}M_{p+1}\frac{|b-a|^{p+1-|\alpha|}}{(p+1-|\alpha|)!} \quad p \in \mathbb{N}, \ |\alpha| \le p, \ \ a,b \in E.$$

We define

$$\mathcal{B}^{\{M\}}(E) := \varinjlim_{\varrho\in\mathbb{N}_{>0}} \mathcal{B}^M_\varrho(E).$$

$F \in \mathcal{B}^{\{M\}}(E)$ is called a *Whitney ultrajet of class* $\mathcal{B}^{\{M\}}$ on E.

Whitney ultrajets III

Let ω be a weight function, $\mathcal{W} = \{W^x : x > 0\}$ the associated weight matrix. A jet F is said to be a *Whitney ultrajet* of class $\mathcal{B}^{\{\omega\}}$ on E if $F \in \mathcal{B}^{\{W^x\}}(E)$ for some $x > 0$.
We set

$$\mathcal{B}^{\{\omega\}}(E) = \mathcal{B}^{\{\mathcal{W}\}}(E) := \varinjlim_{x>0} \mathcal{B}^{\{W^x\}}(E) = \varinjlim_{x>0} \varinjlim_{\varrho>0} \mathcal{B}^{W^x}_{\varrho}(E).$$

This definition coincides with the one given in BBMT '91.

Remarks on condition (mg)

M is said to have *moderate growth* (write (mg)), if

$$\exists\, C \geq 1\ \forall\, j, k \in \mathbb{N} : \quad M_{j+k} \leq C^{j+k} M_j M_k.$$

We know:

(*i*) *ω has $\exists\, H \geq 1\ \forall\, t \geq 0 : \quad 2\omega(t) \leq \omega(Ht) + H$ if and only if some (equivalently each) W^x has (mg).*

(*ii*) *But this happens if and only if $W^x \approx W^y$ for all $x, y > 0$. Hence the classes defined by ω and M do coincide.*

(*iii*) *There exist many equivalent reformulations of (mg), e.g. M has (mg) if and only if*

$$\exists\, C \geq 1\ \forall\, k \in \mathbb{N}_{>0} : \quad \mu_k \leq C(M_k)^{1/k}. \quad (!!!)$$

(*iv*) *For any weight ω the associated matrix $\mathcal{W}$ satisfies generalized/mixed (mg)-conditions (by convexity of φ^*_ω), but the generalization of (iii) is not clear in general!*

Associated weight functions

Let $m=(m_k)\in\mathbb{R}_{>0}^{\mathbb{N}}$ with $m_0=1$ and $m_k^{1/k}\to\infty$ (but not necessarily log-convex). To have this for some/each w^x, $\omega(t)=o(t)$ as $t\to+\infty$ is required.

Consider the well-defined *associated weight function*

$$\omega_m(t):=\sup_{k\in\mathbb{N}}\log\left(\frac{t^k}{m_k}\right),\ t>0,\quad \omega_m(0):=0.$$

The *log-convex minorant* of m is given by

$$\underline{m}_k:=\sup_{t>0}\frac{t^k}{\exp(\omega_m(t))}.$$

Moreover, in this situation, let

$$h_m(t):=\inf_{k\in\mathbb{N}}t^k m_k,\ t>0,\quad h_m(0):=0,$$

i.e. $h_m(t)=\exp(-\omega_m(1/t))$.

Counting functions and their technical problems I

Idea goes back to Dynkin '80 - and then used in Ch./Ch. '94. We define:

$$\overline{\Gamma}_m(t) := \min\{k : h_m(t) = m_k t^k\}, \ t > 0,$$

and, provided that $m_{k+1}/m_k \to \infty$,

$$\underline{\Gamma}_m(t) := \min\left\{k \in \mathbb{N} : \frac{m_{k+1}}{m_k} \geq \frac{1}{t}\right\}, \ t > 0.$$

(*i*) We want to work with sequences w^x (in general they will be NOT log-convex).

(*ii*) If m is log-convex, then $\overline{\Gamma}_m = \underline{\Gamma}_m$ - write Γ_m - (as in Ch./Ch. '94).

(*iii*) Central new idea: We have worked with both counting functions simultaneously.

Counting functions and their technical problems II

We always have $\underline{\Gamma}_m(t) \le \overline{\Gamma}_m(t)$ for all $t > 0$. The proof requires a converse estimate as well - if m is log-convex, then O.K.

Lemma

Let M, N be weight sequences satisfying $m_k^{1/k} \to \infty$, $n_k^{1/k} \to \infty$. Assume that

$$\exists\, C \ge 1\ \forall\, 1 \le j \le k: \quad \frac{\mu_j}{j} \le C \frac{\nu_k}{k}, \tag{7}$$

i.e. property goodness, then

$$\exists\, C \ge 1\ \forall\, t > 0: \quad \overline{\Gamma}_n(Ct) \le \underline{\Gamma}_m(t). \tag{8}$$

Note: In general (8) does not imply (7) (we have a (counter)-example).

Counting functions and their technical problems III

But we will also need:

Lemma

Let M, N and L be weight sequences satisfying $m_k^{1/k} \to \infty$, $n_k^{1/k} \to \infty$ and $l_k^{1/k} \to \infty$ and such that
$\exists\, C \geq 1\ \forall\, 1 \leq j \leq k : \quad \frac{\nu_j}{j} \leq C \frac{\lambda_k}{k}$, *i.e. again property goodness, and*

$$\exists\, C \geq 1\ \forall\, j \in \mathbb{N} : \quad \mu_{2j} \leq C \nu_j,$$

i.e. a mixed moderate growth condition between M and N (in Ch./Ch. '94 also (mg) was heavily used). Then

$$\exists\, D \geq 1\ \forall\, t > 0 : \quad 2\Gamma_l(Dt) \leq \Gamma_m(t).$$

First new result - concavity vs. strong log-convexity I

Proposition

Let σ be a weight function with $\sigma(t) = o(t)$ as $t \to +\infty$ and which is equivalent to a concave weight function. Let $\mathcal{S} = \{S^x : x > 0\}$ be the matrix associated with σ. Then

$$\forall\, x > 0\ \exists\, A, B, C > 0\ \forall\, k \in \mathbb{N} : \quad A^{-1} s_k^{x/B} \leq \underline{s}_k^x \leq s_k^x \leq C^k \underline{s}_k^{Bx}.$$

Moreover we have

$$\exists\, H \geq 1\ \forall\, x > 0\ \forall\, j, k \in \mathbb{N} : \quad \underline{s}_{j+k}^x \leq H^{j+k} \underline{s}_j^{2x} \underline{s}_k^{2x}.$$

Consequence: The matrix $\mathcal{S}$ is equivalent to a matrix $\underline{\mathcal{S}} := \{\underline{S}^x = (k!\underline{s}_k^x)_k : x > 0\}$ consisting only of strongly log-convex sequences and $\underline{\mathcal{S}}$ has the mixed moderate growth condition.

First new result - concavity vs. strong log-convexity II

For the proof of this result we have used several preparations - most important:

(∗) For any weight function σ with $\sigma(t) = o(t)$ as $t \to +\infty$ and which is equiv. to a concave weight function, we have $\omega_{S^x} \sim \omega_{\underline{S}^x}$ for all $x > 0$.

(∗) This holds by $\sigma \sim \omega_{S^x}$ for all $x > 0$ and relating $\omega_{\underline{S}^x}$ to $(\omega^{\iota}_{S^x})_{\star}$, the so-called *lower Legendre conjugate (or envelope)* of $\omega^{\iota}_{S^x}$, writing $\omega^{\iota}_{S^x}(t) := \omega_{S^x}(1/t)$ and $(\omega^{\iota}_{S^x})_{\star}(t) := \inf_{u>0}\{\omega_{S^x}(1/u) + ut\}$.

Another technical problem

...replace in the proof of the main result $\mathcal{S}$ by $\underline{\mathcal{S}}$ to avoid goodnesss....??? - Recall: We also need another mixed moderate growth condition:

$$\forall\, S \in \underline{\mathcal{S}}\ \exists\, T \in \underline{\mathcal{S}}\ \exists\, C \geq 1\ \forall\, k \in \mathbb{N}: \quad \frac{S_{2k}}{S_{2k-1}} \leq C \frac{T_k}{T_{k-1}}.$$

We have been not able to prove directly that $\underline{\mathcal{S}}$ has this property (for $\mathcal{S}$ is follows by convexity of φ_σ^*).

Next new idea: Inspired by a result of Komatsu '73 (characterization of condition (mg) for M) we define for all $x > 0$ and $k \in \mathbb{N}$ a new matrix by

$$v_k^x := \min_{0 \leq j \leq k} \underline{s}_j^{2x} \underline{s}_{k-j}^{2x},$$

and set $\mathcal{V} := \{V^x = (k! v_k^x)_k : x > 0\}$.

Summarize

Proposition

- (∗) *Each* v^x *is log-convex resp. each* V^x *is strongly log-convex.*
- (∗) $2\Gamma_{\underline{s}^{2x}}(t)=\Gamma_{v^x}(t)$ *for all* $x>0$ *and* $t>0$.
- (∗) *There exists* $H\ge 1$ *such that for all* $x>0$ *and* $k\in\mathbb{N}$:

$$\underline{s}^x\le H^k v_k^x\le H^k\underline{s}_k^{2x}.$$

- (∗) $\mathcal{B}^{\{\underline{\mathcal{S}}\}}=\mathcal{B}^{\{\mathcal{S}\}}=\mathcal{B}^{\{\sigma\}}=\mathcal{B}^{\{\mathcal{V}\}}$ *(and* $\mathcal{B}^{(\underline{\mathcal{S}})}=\mathcal{B}^{(\mathcal{S})}=\mathcal{B}^{(\sigma)}=\mathcal{B}^{(\mathcal{V})}$*) as locally convex VS*

Main result - reformulated

Theorem

Let ω be a non-quasianalytic concave weight function.

Let σ be a weight function satisfying $\sigma(t) = o(t)$ as $t \to \infty$. Then the following conditions are equivalent:

(i) *For every compact $E \subseteq \mathbb{R}^n$ the jet mapping $j_E^\infty : \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \to \mathcal{B}^{\{\sigma\}}(E)$ is surjective.*

(ii) *There is $C > 0$ such that $\int_1^\infty \frac{\omega(tu)}{u^2} du \le C\sigma(t) + C$ for all $t > 0$.*

Some comments on the NEW proof of the main result

(∗) Instead of working with $\mathcal{S} = \{S^x : x > 0\}$ directly, and which would require goodness, we are working with (equivalent!) matrices $\underline{\mathcal{S}}$ and $\mathcal{V}$ (so more complicated notation).

(∗) But the idea of working with $\underline{\Gamma}_m$ and $\overline{\Gamma}_m$ becomes superfluous.

(∗) Let $F \in \mathcal{B}^{\{\sigma\}}(E)$ be a Whitney ultrajet, then $F \in \mathcal{B}^{\{V^x\}}(E)$ for some $x > 0$. The extension of F belonging to class $\mathcal{B}^{\{\omega\}}(\mathbb{R}^n)$ will be given by

$$\sum_{i \in \mathbb{N}} \varphi_i(x) \cdot T^{p(x_i)}_{\hat{x}_i} F(x), \quad x \in \mathbb{R}^n \setminus E,$$

φ_i denoting the admissible partition of unity and where the counting function $p(\cdot)$ is defined in terms of $\Gamma_{\underline{S}^{2x}}(\cdot)$.

Consequence 1 - The case $\omega = \sigma$

Corollary

Let ω be a non-quasianalytic weight function, then TFAE:

(i) *For every compact $E \subseteq \mathbb{R}^n$ the jet mapping $j_E^\infty : \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \to \mathcal{B}^{\{\omega\}}(E)$ is surjective.*

(ii) *There is $C > 0$ such that $\int_1^\infty \frac{\omega(tu)}{u^2} du \le C\omega(t) + C$ for all $t > 0$, i.e. ω is a strong weight.*

Consequence 2 - The (mixed) weight sequence case

Theorem

Let M, N be weight sequences with $\mu \leq \nu$ and such that both satisfy (mg)*. Moreover assume that* $\exists\, r > 0\ \exists\, C \geq 1\ \forall\, 1 \leq j \leq k$:

$$\frac{(\mu_j)^{1/r}}{j} \leq C\frac{(\mu_k)^{1/r}}{k}, \quad \frac{(\nu_j)^{1/r}}{j} \leq C\frac{(\nu_k)^{1/r}}{k}$$

and such that $\lim_{j\to+\infty} \frac{(\mu_j)^{1/r}}{j} = +\infty$. *Then the following conditions are equivalent:*

(*i*) *For every compact set* $E \subseteq \mathbb{R}^n$ *the jet mapping* $j^\infty_E : \mathcal{B}_{\{N^{1/r}\}}(\mathbb{R}^n) \to \mathcal{B}_{\{M^{1/r}\}}(E)$ *is surjective.*

(*ii*) $\exists\, C > 0\ \forall\, t \geq 0 : \quad \int_1^\infty \frac{\omega_N(tu)}{u^{1+1/r}}du \leq C\omega_M(t) + C.$

(*iii*) $\sup_{p\in\mathbb{N}_{>0}} \frac{(\mu_p)^{1/r}}{p} \sum_{k\geq p} \left(\frac{1}{\nu_k}\right)^{1/r} < +\infty.$

Notions of goodness

We have called ω good, if $\mathcal{W}$ has the good property and asked:

- $(*)$ Is every concave weight function (equivalent to) a good one?
- $(*)$ Is every strong weight function (equivalent to) a good one?

It is more natural to ask on the weight matrix level for goodness.
ω will be called R-good, if there exists $\mathcal{M}$ such that $\mathcal{B}^{\{\omega\}} = \mathcal{B}^{\{\mathcal{M}\}}$ and

$$\forall\, M \in \mathcal{M}\ \exists\, N \in \mathcal{M}\ \exists\, C \geq 1\ \forall\, 1 \leq j \leq k: \quad \frac{\mu_j}{j} \leq C\frac{\nu_k}{k}.$$

ω will be called B-good, if there exists $\mathcal{M}$ such that $\mathcal{B}^{(\omega)} = \mathcal{B}^{(\mathcal{M})}$ and

$$\forall\, N \in \mathcal{M}\ \exists\, M \in \mathcal{M}\ \exists\, C \geq 1\ \forall\, 1 \leq j \leq k: \quad \frac{\mu_j}{j} \leq C\frac{\nu_k}{k}.$$

Both conditions are mixed almost increasing properties.

Motivated by a result from Javier Jiménez-Garrido and Javier Sanz '16 in the (single) weight sequence situation we can prove:

Lemma

Let $1 = \mu_0 \leq \mu_1 \leq \dots$ and $1 \leq \nu_0 \leq \nu_1 \leq \dots$ be given satisfying $\frac{\mu_j}{j} \leq C\frac{\nu_k}{k}$ for some $C \geq 1$ and all $1 \leq j \leq k$. Then

$$\frac{\tilde{\nu}_k}{k} := \inf_{l \geq k} \frac{\nu_l}{l}, \qquad \tilde{\nu}_0 := 1,$$

satisfies $C^{-1}\mu \leq \tilde{\nu} \leq \nu$ and $k \mapsto \frac{\tilde{\nu}_k}{k}$ is increasing.

Consequence: Each weight matrix which has such a mixed almost increasing property can be replaced by an equivalent one which consists only of strongly log-convex sequences.

Main characterizing theorem

Theorem

Let ω be a weight function with $\omega(t) = o(t)$ as $t \to +\infty$. Then TFAE:

(a) ω *is equivalent to a concave function.*

(b) $\exists\, C \geq 1\ \exists\, t_0 > 0\ \forall\, \lambda \geq 1\ \forall\, t \geq t_0 : \quad \omega(\lambda t) \leq C\lambda\omega(t)$.

(c) $\forall\, x > 0\ \exists\, y > 0\ \exists\, D \geq 1\ \forall\, 1 \leq j \leq k : \quad (w_j^x)^{1/j} \leq D(w_k^y)^{1/k}$.

(d) $\forall\, x > 0\ \exists\, y > 0\ \exists\, D \geq 1\ \forall\, 1 \leq j \leq k : \quad (w_j^y)^{1/j} \leq D(w_k^x)^{1/k}$.

(e) $\mathcal{B}^{\{\omega\}}$ *is stable under composition, under inverse/implicit function theorem and solving ODE's.*

(f) $\mathcal{B}^{(\omega)}$ *is stable under composition, under inverse/implicit function theorem and solving ODE's.*

Main characterizing theorem - continuation

Theorem

Let ω be a weight function with $\omega(t) = o(t)$ as $t \to +\infty$. Then TFAE:

- (a) *ω is equivalent to a concave function.*
- (g) *There does exist a weight matrix $\mathcal{S}$ such that each $S \in \mathcal{S}$ is strongly log-convex and $\mathcal{B}^{\{\omega\}} = \mathcal{B}^{\{\mathcal{S}\}}$.*
- (h) *There does exist a weight matrix $\mathcal{S}$ such that each $S \in \mathcal{S}$ is strongly log-convex and $\mathcal{B}^{(\omega)} = \mathcal{B}^{(\mathcal{S})}$.*
- (i) *ω is R-good.*
- (j) *ω is B-good.*

Currently work in progress: The above conditions are equivalent to the property that $\mathcal{B}^{\{\omega\}}$ resp. $\mathcal{B}^{(\omega)}$ can be described by almost analytic extensions (following and generalizing Dynkin '80).

The weight matrix Roumieu case

Theorem

Let $\mathcal{M} = \{M^x : x > 0\}$ be a weight matrix with $\liminf_k (m_k)^{1/k} > 0$ for all $M \in \mathcal{M}$ and such that for all $M \in \mathcal{M}$ there exists $D \geq 1$ and $N \in \mathcal{M}$ such that $M_{j+1} \leq D^{j+1} N_j$. Consider

(a) $\forall\, M \in \mathcal{M}\ \exists\, N \in \mathcal{M}\ \exists\, C \geq 1\ \forall\, 1 \leq j \leq k: \quad \frac{\mu_j}{j} \leq C \frac{\nu_k}{k}$,

(b) *There does exist $\mathcal{S}$ such that each $S \in \mathcal{S}$ is strongly log-convex and $\mathcal{B}^{\{\mathcal{M}\}} = \mathcal{B}^{\{\mathcal{S}\}}$.*

(c) $\mathcal{B}^{\{\mathcal{M}\}}$ *has the stability properties.*

(d) $\forall\, M \in \mathcal{M}\ \exists\, N \in \mathcal{M}\ \exists\, C \geq 1\ \forall\, 1 \leq j \leq k: \quad (m_j)^{1/j} \leq C(n_k)^{1/k}$.

Then (a) $\Leftrightarrow$ (b) $\Longrightarrow$ (c) $\Leftrightarrow$ (d). We have $\Longleftarrow$, if $\mathcal{M}$ has the strange growth property of Roumieu type.

The weight matrix Beurling case

Theorem

Let $\mathcal{M} = \{M^x : x > 0\}$ *be a weight matrix with* $\lim_k (m_k)^{1/k} = +\infty$ *for all* $M \in \mathcal{M}$ *and such that for all* $M \in \mathcal{M}$ *there exists* $D \geq 1$ *and* $N \in \mathcal{M}$ *such that* $N_{j+1} \leq D^{j+1} M_j$*. Consider*

(a) $\forall\, M \in \mathcal{M}\ \exists\, N \in \mathcal{M}\ \exists\, C \geq 1\ \forall\, 1 \leq j \leq k : \quad \frac{\nu_j}{j} \leq C \frac{\mu_k}{k}$,

(b) *There does exist* $\mathcal{S}$ *such that each* $S \in \mathcal{S}$ *is strongly log-convex and* $\mathcal{B}^{(\mathcal{M})} = \mathcal{B}^{(\mathcal{S})}$.

(c) $\mathcal{B}^{(\mathcal{M})}$ *has the stability properties.*

(d) $\forall\, M \in \mathcal{M}\ \exists\, N \in \mathcal{M}\ \exists\, C \geq 1\ \forall\, 1 \leq j \leq k : \quad (n_j)^{1/j} \leq C(m_k)^{1/k}$.

Then (a) $\Leftrightarrow$ (b) $\Longrightarrow$ (c) $\Leftrightarrow$ (d)*. We have* $\Longleftarrow$*, if* $\mathcal{M}$ *has the strange growth property of Beurling type.*

Some open questions/problems I

(∗) Treat the Beurling case in the mixed setting as well!

Direct proof fails, reduction to the Roumieu case seems to be not easy!

We expect: The main result is also valid for this case (BMT '92 and L. '94 have already shown this but not by a reduction argument).

(∗) Characterize the "strange growth properties" $(\frac{W^x_p}{W^x_{p-1}} \leq A(W^y_p)^{1/p})$ in terms of a (growth) property of ω.

Some open questions/problems II

(∗) We know that $\frac{\mu_j}{j} \le C\frac{\nu_k}{k}$ does imply $\overline{\Gamma}_n(Ct) \le \underline{\Gamma}_m(t)$ and the converse fails in general.

But maybe, having the mixed counting function property, one can change to an equivalent pair of weight sequences $\widetilde{\mu}$ and $\widetilde{\nu}$ satisfying $\frac{\widetilde{\mu}_j}{j} \le C\frac{\widetilde{\nu}_k}{k}$?

We know: In the weight function world one can forget the mixed counting function property (provided ω is equiv. to a concave weight).

If the answer is yes: Then the mixed counting function property is completely superfluous.

If the answer is no: Then in the general weight matrix case one has a better (more general) condition available.

For this recent work see (also the literature citations therein):

Armin Rainer and G. S., On the extension of Whitney ultrajets II, submitted, available online at
`https://arxiv.org/pdf/1808.10253.pdf`.

For the first approach in the mixed setting see:

Armin Rainer and G. S., On the extension of Whitney ultrajets, Studia Math. 245, no. 3, 2019, 255-287, available online at
`https://arxiv.org/pdf/1709.00932.pdf`.