Image domains of locally univalent functions. Part I: Case of univalent functions

Fernando Pérez-González
University of La Laguna

Valencia, March-April 2011

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$.

A univalent function g is a one-to-one holomorphic function in $\mathbb{D}$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$.

A univalent function g is a one-to-one holomorphic function in $\mathbb{D}$.

The class of univalent functions in $\mathbb{D}$ will be denoted by $\mathcal{U}$.

General problems

Question

What do univalent functions belong to some standard space of analytic functions in $\mathbb{D}$*?*

If g is a conformal map from $\mathbb{D}$ onto a Jordan domain Ω whose boundary $\partial\Omega = \mathcal{C}$ is a Jordan curve, If g is a conformal map from $\mathbb{D}$ onto a Jordan domain Ω whose boundary $\partial\Omega = \mathcal{C}$ is a Jordan curve,

Question

A fundamental question in the theory of conformal maps is the relationship between the geometric properties of $\mathcal{C}$ and the analytic properties of g.

Scheme of the talks

- Function spaces and univalence
- Geometric properties
- Local univalence

Hardy spaces

$\mathcal{H}(\mathbb{D})$ is the algebra of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the *Hardy space* H^p consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p}^p := \lim_{r\to 1^-} M_p^p(r,f) := \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p \, dt < \infty.$$

Hardy spaces

$\mathcal{H}(\mathbb{D})$ is the algebra of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the *Hardy space* H^p consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p}^p := \lim_{r\to 1^-} M_p^p(r,f) := \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p \, dt < \infty.$$

The space of bounded analytic functions on $\mathbb{D}$ is denoted by H^∞.

Hardy spaces

$\mathcal{H}(\mathbb{D})$ is the algebra of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the *Hardy space* H^p consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p}^p := \lim_{r\to 1^-} M_p^p(r,f) := \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p \, dt < \infty.$$

The space of bounded analytic functions on $\mathbb{D}$ is denoted by H^∞.

A classical result due to Fatou states that every Hardy function has a radial limit almost everywhere on the unit circle $\mathbb{T} := \{z : |z| = 1\}$.

Hardy spaces

$\mathcal{H}(\mathbb{D})$ is the algebra of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the *Hardy space* H^p consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p}^p := \lim_{r\to 1^-} M_p^p(r,f) := \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt < \infty.$$

The space of bounded analytic functions on $\mathbb{D}$ is denoted by H^∞.

A classical result due to Fatou states that every Hardy function has a radial limit almost everywhere on the unit circle
$\mathbb{T} := \{z : |z| = 1\}$.
$f(\zeta)$ denotes the radial limit of f at $\zeta \in \mathbb{T}$.

$$0 < p < q \Rightarrow H^q \subset H^p$$

Theorem (Prawitz 1927)

If $f \in \mathcal{H}(\mathbb{D})$ is univalent then $f \in H^P$ for all $p < \frac{1}{2}$.

Theorem (Hardy-Littlewood (1920), Pommerenke (1962))

Let $0 < p < \infty$ and suppose $f \in \mathcal{U}$. Then $f \in H^p$ if and only if

$$\int_0^1 M_\infty^p(r,f)dr < \infty\,.$$

Moreover, if $0 < p < 2$ then $f \in H^p$ if and only if

$$\int_0^1 M_1^p(r,f')dr < \infty\,.$$

BMOA

The space *BMOA* of analytic functions with *bounded mean oscillation* on $\mathbb{T}$ consists of those $f \in H^2$ for which

$$\begin{aligned} \|f\|^2_{BMOA} &= \sup_{\zeta \in \mathbb{D}} \|f_\zeta\|^2_{H^2} \\ &= \sup_{\zeta \in \mathbb{D}} \frac{1}{2\pi} \int_{\mathbb{T}} |f(z) - f(\zeta)|^2 \frac{1 - |\zeta|^2}{|z - \zeta|^2} |dz| < \infty, \end{aligned} \tag{2.1}$$

where

$$f_\zeta(z) := (f \circ \varphi_\zeta)(z) - f(\zeta),$$

and

$$\varphi_\zeta(z) := \frac{\zeta - z}{1 - \overline{\zeta} z}.$$

BMOA

The space *BMOA* of analytic functions with *bounded mean oscillation* on $\mathbb{T}$ consists of those $f \in H^2$ for which

$$\begin{aligned} \|f\|^2_{BMOA} &= \sup_{\zeta\in\mathbb{D}} \|f_\zeta\|^2_{H^2} \\ &= \sup_{\zeta\in\mathbb{D}} \frac{1}{2\pi}\int_{\mathbb{T}} |f(z)-f(\zeta)|^2 \frac{1-|\zeta|^2}{|z-\zeta|^2}|dz| < \infty, \end{aligned} \quad (2.1)$$

where

$$f_\zeta(z) := (f\circ\varphi_\zeta)(z) - f(\zeta),$$

and

$$\varphi_\zeta(z) := \frac{\zeta - z}{1-\overline{\zeta}z}.$$

Alternative characterizations of *BMOA*, Garnett's (1981), Baerenstein (1980) or Girela (2000).

VMOA

$$H^\infty \subset BMOA \subset \bigcap_{p>0} H^p \, .$$

The space *VMOA* consists of those $f \in H^2$ for which the integral in (2.1) tends to zero as ζ approaches to the boundary $\mathbb{T}$, i.e

$$\lim_{|z|\to 1^-} \frac{1}{2\pi} \int_{\mathbb{T}} |f(z) - f(\zeta)|^2 \frac{1-|\zeta|^2}{|z-\zeta|^2} |dz| = 0.$$

Bloch space

BMOA is a subspace of the *Bloch space*

$$\mathcal{B} := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty \right\},$$

and *VMOA* is a subspace of both *BMOA* and the *little Bloch space*

$$\mathcal{B}_0 := \left\{ f \in \mathcal{H}(\mathbb{D}) : \lim_{|z|\to 1^-} |f'(z)|(1-|z|^2) = 0 \right\}.$$

It follows from the definition that

$$f \in \mathcal{B} \Rightarrow |f(z)| \lesssim \log \frac{1}{1-|z|^2}, \qquad z \in \mathbb{D}.$$

Dirichlet space

Recall that $f \in \mathcal{H}(\mathbb{D})$ belongs to the classical *Dirichlet space* $\mathcal{D}$ if

$$\|f\|_{\mathcal{D}}^2 := \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) + |f(0)|^2 < \infty,$$

where $dA(z)$ denotes the element of the Lebesgue area measure on $\mathbb{D}$.

Dirichlet space

Recall that $f \in \mathcal{H}(\mathbb{D})$ belongs to the classical *Dirichlet space* $\mathcal{D}$ if

$$\|f\|_{\mathcal{D}}^2 := \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) + |f(0)|^2 < \infty,$$

where $dA(z)$ denotes the element of the Lebesgue area measure on $\mathbb{D}$.

If $f \in \mathcal{D}$, then $Area(f(\mathbb{D}))$ is finite, counting multiplicities.

Dirichlet space

Recall that $f \in \mathcal{H}(\mathbb{D})$ belongs to the classical *Dirichlet space* $\mathcal{D}$ if

$$\|f\|_{\mathcal{D}}^2 := \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) + |f(0)|^2 < \infty,$$

where $dA(z)$ denotes the element of the Lebesgue area measure on $\mathbb{D}$.

If $f \in \mathcal{D}$, then $Area(f(\mathbb{D}))$ is finite, counting multiplicities.

It is easy to see that

$$\mathcal{D} \subset VMOA \subset \mathcal{B}_0.$$

Q_p spaces

Q_p , $0 \leq p < \infty$, is the Möbius invariant subspace of $\mathcal{B}$ that consists of all those functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^2_{Q_p} := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^p(z,a)\,dA(z) < \infty\,.$$

$g(z,a) = -\log|\varphi_a(z)|$ is the Green function of $\mathbb{D}$ with singularity at a. Similarly, we say that $f \in Q_{p,0}$ iff

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^2 g^p(z,a)\,dA(z) = 0.$$

$$Q_{p_1} \subsetneq Q_{p_2}, \qquad 0 \le p_1 < p_2 < \infty.$$

$$Q_0 = \mathcal{D}$$

$$Q_1 = BMOA$$

$$Q_p = \mathcal{B} \qquad p > 1.$$

$$\mathcal{D} \subset Q_{1,0} = VMOA \subset \mathcal{B}_0 = Q_{p,0} \qquad p > 1.$$

Theorem (Pommerenke 1977)

$$BMOA \cap \mathcal{U} = \mathcal{B} \cap \mathcal{U}$$

Theorem (Aulaskari, Lappan, Xiao and Zhao, 1997)

For any $p > 0$,

$$Q_p \cap \mathcal{U} = \mathcal{B} \cap \mathcal{U}$$

Analytic Besov spaces

For $1 < p < \infty$, the *analytic Besov spaces* B^p is defined as the set of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty.$$

Note that $B^2 = \mathcal{D}$.

Analytic Besov spaces

For $1 < p < \infty$, the *analytic Besov spaces* B^p is defined as the set of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty.$$

Note that $B^2 = \mathcal{D}$.

$$B^p \subset \mathcal{B}, \qquad p > 1.$$

Univalent domains

Let X be a space of analytic functions in $\mathbb{D}$

Definition

A planar domain Ω is said to be an X *domain* if every analytic function in $\mathbb{D}$ with the property $f(\mathbb{D}) \subset \Omega$ must belong to X.

Definition

A planar domain Ω will be called *univalent* X *domain* if every $f \in \mathcal{U}$ which applied $\mathbb{D}$ into Ω must belong to X.

If Ω is a simply connected proper domain of the complex plane and $f \in \mathcal{U}$ such that $f(\mathbb{D}) = \Omega$, then

$$d_\Omega(f(z)) \leq |f'(z)|(1 - |z|^2) \leq 4 d_\Omega(f(z)), \quad z \in \mathbb{D},$$

where $d_\Omega(w)$ stands for the Euclidean distance from w to the boundary of Ω.

If Ω is a simply connected proper domain of the complex plane and $f \in \mathcal{U}$ such that $f(\mathbb{D}) = \Omega$, then

$$d_\Omega(f(z)) \leq |f'(z)|(1-|z|^2) \leq 4 d_\Omega(f(z)), \quad z \in \mathbb{D},$$

where $d_\Omega(w)$ stands for the Euclidean distance from w to the boundary of Ω.

Therefore univalent functions in the Bloch space can be characterized by the following well known geometric condition:

If Ω is a simply connected proper domain of the complex plane and $f \in \mathcal{U}$ such that $f(\mathbb{D}) = \Omega$, then

$$d_\Omega(f(z)) \leq |f'(z)|(1 - |z|^2) \leq 4 d_\Omega(f(z)), \quad z \in \mathbb{D},$$

where $d_\Omega(w)$ stands for the Euclidean distance from w to the boundary of Ω.

Therefore univalent functions in the Bloch space can be characterized by the following well known geometric condition:

Theorem

If $f \in \mathcal{U}$ then

$$f \in \mathcal{B} \Leftrightarrow \sup_{w \in \Omega} d_\Omega(w) < \infty,$$

Therefore, the image of $\mathbb{D}$ under f does not contain arbitrarily large discs.

Stegenga described BMOA domain in a quite tangible and visualized way. Put $\Delta(w_0, R) = \{w \in \mathbb{C} : |w - w_0| < R\}$ and $Q(w_0, R) = \Delta(w_0, R) \setminus \Omega$.

Theorem (Stegenga 1978)

Set a domain Ω. The following assertions are equivalent:

(i) *Ω is a BMOA domain.*

(ii) *There exist positive constants R and δ such that*

$$cap(Q(w_0, R)) \geq \delta,$$

for all $w_0 \in \mathbb{C}$.

Stegenga described BMOA domain in a quite tangible and visualized way. Put $\Delta(w_0, R) = \{w \in \mathbb{C} : |w - w_0| < R\}$ and $Q(w_0, R) = \Delta(w_0, R) \setminus \Omega$.

Theorem (Stegenga 1978)

Set a domain Ω. The following assertions are equivalent:

(i) *Ω is a BMOA domain.*

(ii) *There exist positive constants R and δ such that*

$$cap(Q(w_0, R)) \geq \delta,$$

for all $w_0 \in \mathbb{C}$.

The complements of Ω are reasonably thick (measured in potential theory terms) in the vicinity of every point in the plane.

Hedenmalm observed that there is no Q_p domains.

Hedenmalm observed that there is no Q_p domains.

Donaire-Girela- Vukotic (2002) showed that there is no B^p domains.

Theorem (Walsh 2000)

Let $1 < p < \infty$ and let Ω be a simply connected proper domain. If $f \in \mathcal{U}$ and $f(\mathbb{D}) = \Omega$, then $f \in B^p$ if and only if

$$\int_\Omega d_\Omega(w)^{p-2} dA(w) < \infty.$$

P-G and Rättyä

For $0 < p < \infty$ and $-1 < \alpha < \infty$, the *weighted Bergman space* A^p_α consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\alpha} := \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty,$$

Theorem (Baernstein-Girela- Peláez 2007)

Let $0 < p < \infty$, $-1 < \alpha < \infty$ and $f \in \mathcal{U}$. Then $f \in A^p_\alpha$ if and only if

$$\int_0^1 r(1-r^2)^\alpha \left(\int_0^r M^p_\infty(\rho, f) \, d\rho \right) dr < \infty.$$

Moreover, if $0 < p < 2$, then $f \in A^p_\alpha$ if and only if

$$\int_0^1 r(1-r^2)^\alpha \left(\int_0^r M^p_1(\rho, f') \, d\rho \right) dr < \infty.$$

The Hardy space H^p is identified with the limit space of the weighted Bergman space A^p_α as $\alpha \to -1^+$, and therefore the notation $A^p_{-1} := H^p$ is adopted.

The Hardy space H^p is identified with the limit space of the weighted Bergman space A^p_α as $\alpha \to -1^+$, and therefore the notation $A^p_{-1} := H^p$ is adopted.

One reason to do so is that $\lim_{\alpha \to -1^+} \|f\|_{A^p_\alpha} = \|f\|_{H^p}$

The Hardy space H^p is identified with the limit space of the weighted Bergman space A^p_α as $\alpha \to -1^+$, and therefore the notation $A^p_{-1} := H^p$ is adopted.

One reason to do so is that $\lim_{\alpha \to -1^+} \|f\|_{A^p_\alpha} = \|f\|_{H^p}$

On the other hand, a generalization of the Littlewood-Paley formula due to Stein states that

$$\|f\|^p_{H^p} = \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} \, dA(z) + |f(0)|^p \,. \quad (2.2)$$

An analogous formula for the weighted Bergman space exists, namely

$$\|f\|_{A^p_\alpha}^p \simeq \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \left(\log \frac{1}{|z|}\right)^{\alpha+2} dA(z) + |f(0)|^p \,, \quad (2.3)$$

(Smith, 1996)

In Theorem of Baernstein et al., $\alpha = -1$ can not be substituted since the singularities would become too strong. However, an application of Fubini's theorem yields

$$\int_0^1 r(1-r^2)^\alpha \left(\int_0^r M_\infty^p(\rho,f)\,d\rho\right)\,dr$$

$$= \int_0^1 M_\infty^p(\rho,f)\int_\rho^1 r(1-r^2)^\alpha\,dr\,d\rho$$

$$= \frac{1}{2(\alpha+1)}\int_0^1 M_\infty^p(\rho,f)(1-\rho^2)^{\alpha+1}\,d\rho,$$

and similarly

$$\int_0^1 r(1-r^2)^\alpha \left(\int_0^r M_1^p(\rho,f')\,d\rho\right)\,dr$$

$$= \frac{1}{2(\alpha+1)}\int_0^1 M_1^p(\rho,f')(1-\rho^2)^{\alpha+1}\,d\rho.$$

This shows that Theorem of Baernstein, Girela and Peláez indeed generalizes Theorem of Hardy-Littlewood et al. for the weighted Bergman spaces, and thus the following result holds.

Theorem

Let $0 < p < \infty$, $-1 \leq \alpha < \infty$ and $f \in \mathcal{U}$. Then $f \in A^p_\alpha$ if and only if

$$J^p_\alpha(f) := \int_0^1 M^p_\infty(r, f)(1 - r^2)^{\alpha+1}\, dr < \infty.$$

Moreover, if $0 < p < 2$, then $f \in A^p_\alpha$ if and only if

$$K^p_\alpha(f) := \int_0^1 M^p_1(r, f')(1 - r^2)^{\alpha+1}\, dr < \infty.$$

The second part of the assertion is true in the case $p = 1$ and $-1 < \alpha < \infty$ for all $f \in \mathcal{H}(\mathbb{D})$. To see this, it suffices to observe that

$$\|f\|_{A^1_\alpha} \simeq \|f'\|_{A^1_{\alpha+1}} + |f(0)| \simeq K^1_\alpha(f) + |f(0)|,$$

where the first asymptotic equality follows by the well-known result $\|f\|_{A^p_\alpha} \simeq \|f'\|_{A^p_{p+\alpha}} + |f(0)|$ for all $0 < p < \infty$ and $-1 < \alpha < \infty$, and the second one is a simple consequence of the fact that $M^p_1(r, f')$ is an increasing function of r.

Hardy-type spaces

For $0 < p < \infty$ and $-3 < \alpha < \infty$, the **Hardy type space** H^p_α consists of those analytic functions f in $\mathbb{D}$ for which

$$\|f\|^p_{H^p_\alpha} := \int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^{\alpha+2} \, dA(z) < \infty.$$

This function space appears in several occasions in the existing literature.

- Mateljević and Pavlović (1983)
- Girela, Pavlović and Peláez (2007)

For $0 < p < \infty$ and $-2 < \alpha < \infty$, the S^p_α consists of those analytic functions f in $\mathbb{D}$ for which

$$\|f\|^p_{S^p_\alpha} := \int_0^1 r(1-r^2)^{\alpha+1}\left(\int_{\Delta(0,r)} |f'(z)|^2\, dA(z)\right)^{\frac{p}{2}} dr < \infty.$$

Theorem (P.-G. and Rättyä 2008)

Let $0 < p < \infty$ *and* $-2 \leq \alpha < \infty$. *Then*

$$H^p_\alpha \cap \mathcal{U} = S^p_\alpha \cap \mathcal{U}\,.$$

Theorem (P.-G. and Rättyä 2008)

Let $0 < p < \infty$ *and* $-2 \leq \alpha < \infty$ *and suppose* $f \in \mathcal{U}$. *The following assertions are equivalent*

- $f \in \mathcal{B}$.
- $\sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^p_\alpha}$.
- $\sup_{a\in\mathbb{D}} \|f \circ \varphi_a\|_{S^p_\alpha}$.
- $\sup_{a\in\mathbb{D}} J^p_\alpha(f \circ \varphi_a - f(a)) < \infty$.

In 1978, Pommerenke characterized both asymptotically conformal and asymptotically smooth curves in terms of analytic properties of $\log g'$.

In 1978, Pommerenke characterized both asymptotically conformal and asymptotically smooth curves in terms of analytic properties of $\log g'$.

Let $\mathcal{C}$ be a (closed) Jordan curve in the complex plane $\mathbb{C}$ and let $C(w_I, w_2)$ denote the smaller arc of $\mathcal{C}$ between the points w_1 and w_2 on $\mathcal{C}$.

In 1978, Pommerenke characterized both asymptotically conformal and asymptotically smooth curves in terms of analytic properties of $\log g'$.

Let $\mathcal{C}$ be a (closed) Jordan curve in the complex plane $\mathbb{C}$ and let $C(w_l, w_2)$ denote the smaller arc of $\mathcal{C}$ between the points w_1 and w_2 on $\mathcal{C}$.

$\mathcal{C}$ is called *asymptotically conformal* if

$$\max_{w\in\mathcal{C}(w_1,w_2)} \frac{|w_2 - w| + |w - w_1|}{|w_2 - w_1|} \to 1, \quad \text{as} \quad |w_2 - w_1| \to 0,$$

In 1978, Pommerenke characterized both asymptotically conformal and asymptotically smooth curves in terms of analytic properties of $\log g'$.

Let $\mathcal{C}$ be a (closed) Jordan curve in the complex plane $\mathbb{C}$ and let $C(w_l, w_2)$ denote the smaller arc of $\mathcal{C}$ between the points w_1 and w_2 on $\mathcal{C}$.

$\mathcal{C}$ is called *asymptotically conformal* if

$$\max_{w\in\mathcal{C}(w_1,w_2)} \frac{|w_2 - w| + |w - w_1|}{|w_2 - w_1|} \to 1, \quad \text{as} \quad |w_2 - w_1| \to 0,$$

and *quasi-conformal* if this maximum is uniformly bounded for all $w_1, w_2 \in \mathcal{C}$.

asymptotically conformal

asymptotically conformal

The latter case occurs if and only if $\mathcal{C}$ is the image of a circle under a *quasi-conformal mapping* of $\mathbb{C}$ (Pommerenke, 1975), and therefore quasi-conformal curves are usually called *quasi-circles*.

asymptotically conformal

The latter case occurs if and only if $\mathcal{C}$ is the image of a circle under a *quasi-conformal mapping* of $\mathbb{C}$ (Pommerenke, 1975), and therefore quasi-conformal curves are usually called *quasi-circles*.

The concept of asymptotically conformal curves was introduced by Becker in 1972 and it fits nicely between the theories of quasi-conformal and smooth curves.

asymptotically smooth

In addition, recall that if $\mathcal{C}$ is a rectifiable Jordan curve and $l(w_1, w_2)$ denotes the length of the shorter arc on $\mathcal{C}$ joining w_1 and w_2, then $\mathcal{C}$ is said to be *asymptotically smooth* if

$$\frac{l(w_1, w_2)}{|w_2 - w_1|} \to 1, \quad \text{as} \quad |w_2 - w_1| \to 0,$$

and *quasi smooth* if this quotient is uniformly bounded for all $w_1, w_2 \in \mathcal{C}$.

asymptotically smooth

In addition, recall that if $\mathcal{C}$ is a rectifiable Jordan curve and $l(w_1, w_2)$ denotes the length of the shorter arc on $\mathcal{C}$ joining w_1 and w_2, then $\mathcal{C}$ is said to be *asymptotically smooth* if

$$\frac{l(w_1, w_2)}{|w_2 - w_1|} \to 1, \quad \text{as} \quad |w_2 - w_1| \to 0,$$

and *quasi smooth* if this quotient is uniformly bounded for all $w_1, w_2 \in \mathcal{C}$.

Inner domains of quasi smooth curves are also known as *chord-arc* or *Lavrentiev domains.*

Theorems of Pommerenke 1978

Let g be a conformal map from the unit disc $\mathbb{D}$ onto the inner domain bounded by the Jordan curve $\mathcal{C} = g(\mathbb{T})$.

Theorems of Pommerenke 1978

Let g be a conformal map from the unit disc $\mathbb{D}$ onto the inner domain bounded by the Jordan curve $\mathcal{C} = g(\mathbb{T})$.

- $\mathcal{C}$ is asymptotically conformal if and only if $\log g'$ belongs to the *little Bloch space* $\mathcal{B}_0$.

Theorems of Pommerenke 1978

Let g be a conformal map from the unit disc $\mathbb{D}$ onto the inner domain bounded by the Jordan curve $\mathcal{C} = g(\mathbb{T})$.

- $\mathcal{C}$ is asymptotically conformal if and only if $\log g'$ belongs to the *little Bloch space* $\mathcal{B}_0$.
- $\mathcal{C}$ is asymptotically smooth if and only if $\log g'$ belongs to *VMOA*.

$\log g' \in BMOA$

For g locally univalent in $\mathbb{D}$ ($g'(z) \neq 0$ for any z), the *Schwarzian derivative* is defined as

$$S_g(z) = \left(\frac{g''(z)}{g'(z)}\right)' - \frac{1}{2}\left(\frac{g''(z)}{g'(z)}\right)^2$$

$\log g' \in BMOA$

For g locally univalent in $\mathbb{D}$ ($g'(z) \neq 0$ for any z), the *Schwarzian derivative* is defined as

$$S_g(z) = \left(\frac{g''(z)}{g'(z)}\right)' - \frac{1}{2}\left(\frac{g''(z)}{g'(z)}\right)^2$$

Astala and Zeinsmeister (1991) studied the set of conformal maps g such that $\log g' \in BMOA$.

$\log g' \in BMOA$

For g locally univalent in $\mathbb{D}$ ($g'(z) \neq 0$ for any z), the *Schwarzian derivative* is defined as

$$S_g(z) = \left(\frac{g''(z)}{g'(z)}\right)' - \frac{1}{2}\left(\frac{g''(z)}{g'(z)}\right)^2$$

Astala and Zeinsmeister (1991) studied the set of conformal maps g such that $\log g' \in BMOA$.

Theorem (Astala-Zinsmeister 1991)

Let f be a conformal map on $\mathbb{D}$. Then the following assertions are equivalent:

- $\log g' \in BMOA$ *with small norm.*
- $g(\partial\mathbb{D})$ *is a Lavrentiev curve.*
- $|S_g(z)|^2(1-|z|^2)^3 dA(z)$ *is a Carleson measure on* $\mathbb{D}$.

Bishop and Jones obtained a complete (analytic and geometric) description of those simply connected domains Ω such that any Riemann map g of $\mathbb{D}$ onto Ω satisfies $\log g' \in BMOA$.

Bishop and Jones obtained a complete (analytic and geometric) description of those simply connected domains Ω such that any Riemann map g of $\mathbb{D}$ onto Ω satisfies $\log g' \in BMOA$.

Theorem (Bishop-Jones 1994)

The following assertions are equivalent:

- $\log g' \in BMOA$.
- $|S_g(z)|^2(1-|z|^2)^3 dA(z)$ *is a Carleson measure on* $\mathbb{D}$.
- *There exist* $\delta > 0$ *and* $C > 0$ *such that for all* $z_0 \in \Omega$ *there is a subdomain* $U \subset \Omega$ *such that*
  - (i) $z_0 \in U$ *and* $dist(z_0, \partial\Omega) \leq dist(z_0, \partial U)$.
  - (ii) ∂U *is chord-arc with constant at most* C *and* $l(\partial U) \leq C dist(z_0, \partial\Omega)$.
- *There exist* $\delta > 0$ *and* $C > 0$ *such that for all* $z_0 \in \Omega$ *there is a Lipschitz domain* $V \subset \mathbb{D}$ *such that*
  - (i) $z_0 \in V$.
  - (ii) $\omega(z_0, \partial V \cap \partial\mathbb{D}, V) \geq \delta$, *and*
  - (iii) $\int_V |g'(z)||S_g(z)|^2(1-|z|^2)^3 dA(z) \leq C|g'(z_0)|(1-|z_0|^2)$.

These results in terms of the Schwarzian derivative have been recently extended to other spaces of functions

Theorem (Pau-Peláez 2009)

For $0 < p < \infty$, the following assertions are equivalent:

- $\log g' \in Q_p$.
- $|S_g(z)|^2(1-|z|^2)^{p+2}dA(z)$ *is a Carleson measure on* $\mathbb{D}$.

If $\Gamma = \partial\Omega$ is a Jordan curve and g is a conformal map from $\mathbb{D}$ onto Ω, let us consider the geometric quantity

$$\eta(\delta) := \sup_{|w_1 - w_2| \leq \delta} \sup_{w \in \Gamma(w_1, w_2)} \left(\frac{|w_2 - w| + |w - w_1|}{|w_2 - w_1|} - 1 \right)^{\frac{1}{2}}, \quad 0 \leq \delta < 1.$$

Theorem (Pau-Peláez)

Let $0 < p < 1$ and let $g \in \mathcal{U}$ such that $\Gamma = \partial g(\mathbb{D})$ is a closed Jordan curve. If

$$\int_0^1 \frac{\eta^2(t)}{t^{2-p}} < \infty$$

then $\log g' \in Q_{p,0}$

Theorem (PG- Rättyä 2009)

Let $1 < p < \infty$ and $g : \mathbb{D} \to \Omega$ be a conformal map such that $g(\mathbb{T})$ is a closed Jordan curve. Then, the following assertion are equivalent:

- $\log g' \in B_p$
- $$I(g) := \int_{\mathbb{D}} |S_g(z)|^p (1-|z|^2)^{2p-2} dA(z) < \infty .$$

In particular, $\log g' \in \mathcal{D}$ *if and only if* $S_g(z)(1-|z|^2) \in L^2(\mathbb{D})$.

Theorem (PG- Rättyä 2009)

Let $0 < s \leq 1$ *and* $g : \mathbb{D} \to \Omega$ *be a conformal map such that* $g(\mathbb{T})$ *is a closed Jordan curve. Then, the following assertion are equivalent:*

- $\log f' \in Q_{s,0}$
- $|S_g(z)|^2(1-|z|^2)^{2+s}dA(z)$ *is a vanishing* $s-$ *Careson measure.*

In particular,

- $\log g' \in VMOA$ *if and only if* $|S_g(z)|^2(1-|z|^2)^3 dA(z)$ *is a vanishing Carleson measure on* $\mathbb{D}$*.*

These results have been recently extended to the general class of spaces, the so-called $F_{p,q,s}$, where $p > 0$, $q > -2$ and $s \geq 0$, defined as the set of all analytic functions f in $\mathbb{D}$ for which

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) < \infty\,.$$

- $F_{p,p-2,0} = B^p$.
- $F_{2,0,0} = \mathcal{D}$
- $F_{2,0,s} = Q_p$
- $F_{2,0,1} = BMOA$

For $q+s \geq -1$, $1 \leq p < \infty$ and $q+2 \leq p$, then $F_{p,q,s} \subset \mathcal{B}$.

Theorem (Zorboska 2011)

Let $1 \leq p < \infty$, $-2 < q < \infty$, $0 \leq s$ and $q + s > -1$ and $f \in \mathcal{U}$.

- *For $p = q + 2$, $f \in F_{p,q,s}$ if and only if $d\mu_(z) = |S_f(z)|^p(1 - |z|^2)^{p+q+s} dA(z)$ is an s-Carleson measure on $\mathbb{D}$.*
- *For $p > q + 2$, $f \in F_{p,q,s}$ if and only if $\log f' \in \mathcal{B}_0$ and $d\mu_(z) = |S_f(z)|^p(1 - |z|^2)^{p+q+s} dA(z)$ is an s-Carleson measure on $\mathbb{D}$.*

THANK YOU VERY MUCH