Lipschitz subspaces of $C(K)$

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- $(C(K), \|\cdot\|)$ is a separable Banach space.
- $\mathbb{I} := [0, 1]$.

Theorem (Banach-Mazur)

$C(\mathbb{I})$ is a universal Banach space for the class of all separable Banach spaces. Namely, if X is a separable Banach space, then there exists a linear isometric embedding

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Theorem

Let Δ be the Cantor set. Then $C(\Delta)$ is a universal Banach space for the class of all separable Banach spaces.

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If K is a noncountable compact metric space, there exists $H \subset K$ homeomorphic to Δ.

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If K_1 and K_2 are uncountable metrizable compacta, then $C(K_1)$ contains an isometric copy of $C(K_2)$ and vice versa.

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It is not possible to obtain topological information about K from topological or linear information of $C(K)$.

We will show how to obtain some information about K from geometric information of $C(K)$.

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Answer: as the linear subspace of all constant maps.

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Answer: we need to find two suitable maps f_1 and f_2 in $C(\mathbb{I})$ in order to send an orthonormal base of $\mathbb{R}^2$ into $\{f_1, f_2\}$.

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Thus we can define $j : \mathbb{R}^2 \to C(\mathbb{I})$ as

$$j(a,b)(t) = a\cos(\pi t) + b\sin(\pi t)$$

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\begin{aligned}
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Homework: Find a linear and isometric embedding of $(\mathbb{R}^2, \|\cdot\|_1)$ and $(\mathbb{R}^2, \|\cdot\|_\infty)$ into $C(\mathbb{I})$.

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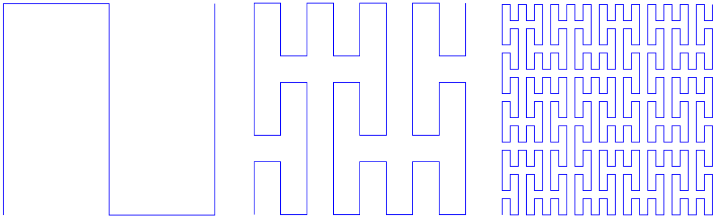
Special homework for super clever students: Find an isometric and linear embedding of the euclidean space $\mathbb{R}^3$ into $C(\mathbb{I})$.

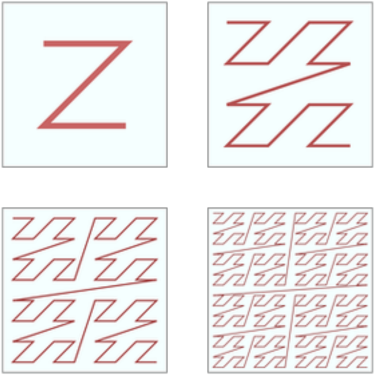
Open parenthesis: Peano curves

Definition

A peano curve or a space filling curve is a continuous onto map

$$f : \mathbb{I} \mapsto \mathbb{I}^n.$$





Theorem (W. F. Donogue, 1957)

Let $H \subset C(\mathbb{I})$ *be a subspace isometrically isomorphic to a Hilbert space. If* $k < \dim H$ *and* $\{e_1, \dots, e_k\}$ *is a linearly independent subset of* H*, then* $\gamma : \mathbb{I} \to \mathbb{R}^k$ *given by*

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is a space filling curve.

If we embed $\mathbb{R}^3$ into $C(\mathbb{I})$, then we can get a formula for a Peano curve

$$\gamma : \mathbb{I} \to \mathbb{I}^2.$$

Close parenthesis

$C(\mathbb{I}^2)$ is a Universal Banach space too. Can we find an embedding of the euclidian space $\mathbb{R}^3$ into $C(\mathbb{I}^2)$?

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$$j(a, b, c)(t, s) = a \cos(\pi t) \cos(\pi s) + b \sin(\pi t) \cos(\pi s) + c \sin(\pi s).$$

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What about $\mathbb{R}^n$ for $n \geq 4$?

Theorem (M. Raja, N. J-P)

Let (K, d) be an uncountable metric compact space and $n \in \mathbb{N}$. The following are equivalent:

(i) *There is an onto Lipschitz mapping $\phi : K \to \mathbb{I}^n$.*

(ii) *$C(K)$ contains an isometric copy of any $(n+1)$-dimensional Banach space made of Lipschitz functions.*

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Corollary

In $C(\mathbb{I}^k)$ we can find an isometric copy of any $(n+1)$-dimensional Banach space made of Lipschitz functions if and only if $k \geq n$.

For X and Y metric spaces, if $f : X \to Y$ is a Lipschitz map, then $\dim_H(X) \geq \dim_H(f(X))$.

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Corollary

Let (K, d) be an uncountable metric compact space and $n \in \mathbb{N}$. If $C(K)$ contains an isometric copy of any $(n+1)$-dimensional Banach space made of Lipschitz functions, then

$$\dim_H(K) \geq n$$

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Theorem (T. Keleti, A. Mathé, O. Zindulka)

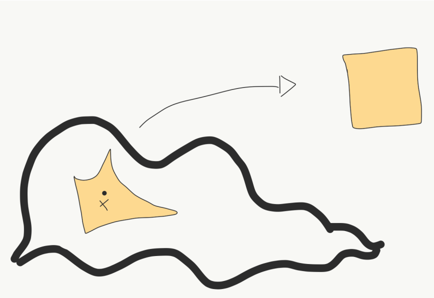
Let K be a compact metric space with $\dim_H(K) > n$. Then K can be mapped onto $\mathbb{I}^n$ by a Lipschitz map.

Corollary

Let K be a compact metric space with $\dim_H(K) > n$. Then $C(K)$ contains an isometric copy of any $(n+1)$-dimensional Banach space made of Lipschitz functions.

Definition

A separable metric space is called a *Lipschitz manifold* (of dimension n) if every point has a closed neighborhood which is Lipschitz homeomorphic to $\mathbb{I}^n$, that is, there is a Lipschitz bijective mapping whose inverse is Lipschitz too.



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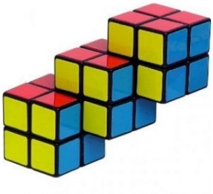
- 

Theorem (M. Raja-N. J-P.)

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Moreover, if K is a Lipschitz manifold, then statements (i), (ii) and (iii) are also equivalent to

(iv) *The dimension of K is at least n.*

Theorem (M. Raja-N. J-P.)

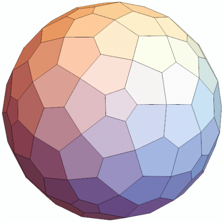
Let K be a compact C^r-manifold of dimension $n-1$ for $n \geq 2$ and $r = 1, \ldots, \infty$. Then

(a) *$C(K)$ contains an isometric copy of $(\mathbb{R}^n, \|\cdot\|_2)$ made of C^r-smooth functions;*

(b) *$C(K)$ contains no isometric copy of $(\mathbb{R}^{n+1}, \|\cdot\|_2)$ made of C^1-smooth functions or Lipschitz functions.*

Theorem (M. Raja-N. J-P.)

If K is an infinite metric compact space, then $C(K)$ contains isometric copies made of Lipschitz functions of any finite dimensional polyhedral space.



Open problem

Can we define a less geometric and more topological notion of dimension, in such a way that this kind of dimension is characterized by the way we can embed any n-dimensional Banach space in $C(K)$?

Final observations

- The Banach Mazur distance between two isomorphic Banach spaces is defined as

$$d_{BM}(X, Y) = \ln\left(\inf\{\|T\|\|T^{-1}\| \ : \ T : X \to Y \text{ is an isomorphism}\}\right)$$

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- Let us denote by $BM(n)$ the set of isometry classes of n-dimensional Banach spaces, equipped with the Banach-Mazur distance.
- The subset of $BM(n)$ corresponding with all strictly convex and smooth norms is dense in $BM(n)$.

- Given X and Y linear subspaces of $C(\mathbb{I})$, let us denote by B_X and B_Y their closed unitary balls. Define $g(X, Y) = d_H(B_X, B_Y)$.
- $g(X, Y)$ is known as the Gokhberg-Markus gap.
- Let $GM(n)$ denote the space of all n-dimensional subspaces of $C(\mathbb{I})$ equipped with the distance g.

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- The family of all polyhedral subspaces is dense in $GM(n)$.
- The set of all strictly convex and smooth subspaces is not dense for $n \geq 2$.

- W. F. Donoghue, *Continuous function spaces isometric to a Hilbert space*, Proc. Amer. Math. Soc. 8 (1957), 1–2.
- M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis.* CMS Books in Mathematics, Springer, New York, 2011.
- N. Jonard-Pérez, M. Raja Baños, *Lipschitz subspaces of C(K).* Topology and its Applications, 204 (2016) 149-156.

Thank you for your attention