On interpolation of analytic families of multilinear operators

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Outline

- Stein's interpolation theorem [Trans. Amer. Math. Soc. (1956)] for analytic families of operators between L^p spaces $(p \geqslant 1)$ has found several significant applications in harmonic analysis. This theorem provides a generalization of the classical single-operator Riesz-Thorin interpolation theorem to a family $\{T_z\}$ of operators that depend analytically on a complex variable z.
- In the framework of Banach spaces, interpolation for analytic families of multilinear operators can be obtained via duality in a way similar to that used in the linear case. For instance, one may adapt the proofs in Zygmund book and Berg and Löfstrom for a single multilinear operator to a family of multilinear operators. However, this duality-based approach is not applicable to quasi-Banach spaces since their topological dual spaces may be trivial.

The open strip $\{z;\, 0 < \mathrm{Re}\, z < 1\}$ in the complex plane is denoted by S, its closure by $\overline{S}$ and its boundary by ∂S.

Definition Let $A(S)$ be the space of scalar-valued functions, analytic in S and continuous and bounded in $\overline{S}$. For a given couple (A_0, A_1) of quasi-Banach spaces and A another quasi-Banach space satisfying $A \subset A_0 \cap A_1$, we denote by $\mathcal{F}(A)$ the space of all functions $f\colon S \to A$ that can be written as finite sums of the form

$$f(z) = \sum_{k=1}^{N} \varphi_k(z) a_k, \quad z \in \overline{S},$$

where $a_k \in A$ and $\varphi_k \in A(S)$. For every $f \in \mathcal{F}(A)$ we set

$$\|f\|_{\mathcal{F}(A)} = \max\Big\{\sup_{t\in\mathbb{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{A_1}\Big\}.$$

Remark Clearly we have that $\|a\|_\theta \leqslant \|a\|_{A_0\cap A_1}$ for every $a \in A_0 \cap A_1$, and notice that $\|\cdot\|_\theta$ could be identically zero.

Definition A quasi-Banach couple is said to be admissible whenever $\|\cdot\|_\theta$ is a quasi-norm on $A_0 \cap A_1$, and in this case, the quasi-normed space $(A_0 \cap A_1, \|\cdot\|_\theta)$ is denoted by $(A_0, A_1)_\theta$.

Remark If A is dense in $A_0 \cap A_1$, then for every $a \in A$ we have

$$\|a\|_\theta = \inf\{\|f\|_{\mathcal{F}(A)};\ f \in \mathcal{F}(A),\ f(\theta) = a\}.$$

Definition If there is a completion of $(A_0, A_1)_\theta$ which is set-theoretically contained in $A_0 + A_1$, then it is denoted by $[A_0, A_1]_\theta$.

Theorem [A. P. Calderón, Studia Math. (1964)] If (A_0, A_1) is a Banach couple, then for every $f \in \mathcal{F}(A_0 \cap A_1)$, and $0 < \theta < 1$,

$$\log \|f(\theta)\|_\theta \leqslant \int_{-\infty}^{\infty} \log \|f(it)\|_{A_0} P_0(\theta, t)\, dt + \int_{-\infty}^{\infty} \log \|f(1+it)\|_{A_1} P_1(\theta, t)\, dt,$$

where $P_0(\theta, t)$ and $P_1(\theta, t)$ are the values of the Poisson kernels of the strip, on $\mathrm{Re}\, z = 0$ and $\mathrm{Re}\, z = 1$, respectively.

Remark The same estimate holds in the case of quasi-Banach spaces; the proof is similar as in the Banach case.

- The Poisson kernels P_j $(j = 0, 1)$ for the strip are given by

$$P_j(x+iy, t) = \frac{e^{-\pi(t-y)} \sin \pi x}{\sin^2 \pi x + (\cos \pi x - (-1)^j e^{-\pi(t-y)})^2}, \quad x + iy \in \overline{S}.$$

Using the fact that the Poisson kernels satisfy

$$\int_{\mathbb{R}} P_0(\theta, t)\, dt = 1 - \theta, \qquad \int_{\mathbb{R}} P_1(\theta, t)\, dt = \theta$$

together with Calderón's inequality and Jensen's inequality, and the concavity of the logarithmic function, we obtain the following result:

Lemma Let (A_0, A_1) be a couple of complex quasi-Banach spaces. For every f in $\mathcal{F}(A_0 \cap A_1)$, $0 < p_0, p_1 < \infty$, and $0 < \theta < 1$ we have

$$\|f(\theta)\|_\theta \leqslant \Big(\frac{1}{1-\theta} \int_{-\infty}^{\infty} \|f(it)\|_{A_0}^{p_0} P_0(\theta, t)\, dt\Big)^{\frac{1-\theta}{p_0}} \Big(\frac{1}{\theta} \int_{-\infty}^{\infty} \|f(1+it)\|_{A_1}^{p_1} P_1(\theta, t)\, dt\Big)^{\frac{\theta}{p_1}}$$

Definition A continuous function $F\colon \overline{S} \to \mathbb{C}$ which is analytic in S is said to be of admissible growth if there is $0 \leqslant \alpha < \pi$ such that

$$\sup_{z\in\overline{S}} \frac{\log|F(z)|}{e^{\alpha|\mathrm{Im}\, z|}} < \infty.$$

Lemma [I. I. Hirchman, J. Analyse Math. (1953)] If a function $F\colon \overline{S} \to \mathbb{C}$ is analytic, continuous on $\overline{S}$, and is of admissible growth, then

$$\log|F(\theta)| \leqslant \int_{-\infty}^{\infty} \log|F(it)|\, P_0(\theta,t)\, dt + \int_{-\infty}^{\infty} \log|F(1+it)|\, P_1(\theta,t)\, dt.$$

Definition Let (Ω, Σ, μ) be a measure space and let $\mathcal{X}_1,\dots,\mathcal{X}_m$ be linear spaces. The family $\{T_z\}_{z\in\overline{S}}$ of multilinear operators $T\colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to \widetilde{L}^0(\mu)$ is said to be analytic if for any $(x_1,\dots,x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$ and for almost every $\omega \in \Omega$ the function

$$z \mapsto T_z(x_1,\dots,x_m)(\omega), \quad z \in \overline{S}$$

is analytic in S and continuous on $\overline{S}$. Additionally, if for $j = 0$ and $j = 1$ the function

$$(t,\omega) \mapsto T_{j+it}(x_1,\dots,x_m)(\omega), \quad (t,\omega) \in \mathbb{R} \times \Omega \tag{$*$}$$

is $(\mathcal{L} \times \Sigma)$-measurable for every $(x_1,\dots,x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, and for almost every $\omega \in \Omega$ the function given by formula $(*)$ is of admissible growth, then the family $\{T_z\}$ is said to be an admissible analytic family. Here $\mathcal{L}$ is the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$.

- A quasi-Banach lattice X is said to be maximal whenever $0 \leqslant f_n \uparrow f$ a.e., $f_n \in X$, and $\sup_{n\geqslant 1} \|f_n\|_X < \infty$ implies that $f \in X$ and $\|f_n\|_X \to \|f\|_X$.
- A quasi-Banach lattice X is said to be p-convex $(0 < p < \infty)$ if there exists a constant $C > 0$ such that for any $f_1,\dots,f_n \in X$ we have

$$\Big\|\Big(\sum_{k=1}^{n} |f_k|^p\Big)^{1/p}\Big\|_X \leqslant C \Big(\sum_{k=1}^{n} \|f_k\|_X^p\Big)^{1/p}.$$

  The optimal constant C in this inequality is called the p-convexity constant of X, and is denoted, by $M^{(p)}(X)$.
- A quasi-Banach lattice is said to have nontrivial convexity whenever it is p-convex for some $0 < p < \infty$.

Theorem For each $1 \leqslant i \leqslant m$, let $\overline{X}_i = (X_{0i}, X_{1i})$ be admissible couples of quasi-Banach spaces, and let (Y_0, Y_1) be a couple of maximal quasi-Banach lattices on a measure space (Ω, Σ, μ) such that each Y_j is p_j-convex for $j = 0, 1$. Assume that $\mathcal{X}_i$ is a dense linear subspace of $X_{0i} \cap X_{1i}$ for each $1 \leqslant i \leqslant m$, and that $\{T_z\}_{z \in \overline{S}}$ is an admissible analytic family of multilinear operators $T_z \colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to Y_0 \cap Y_1$. Suppose that for every $(x_1, \dots, x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $t \in \mathbb{R}$ and $j = 0, 1$,

$$\|T_{j+it}(x_1, \dots, x_m)\|_{Y_j} \leqslant K_j(t)\|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}},$$

where K_j are Lebesgue measurable functions such that $K_j \in L^{p_j}(P_j(\theta, \cdot)\, dt)$ for all $\theta \in (0,1)$. Then for all $(x_1, \dots, x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, all $s \in \mathbb{R}$, and all $0 < \theta < 1$ we have

$$\|T_{\theta+is}(x_1, \dots, x_m)\|_{Y_0^{1-\theta} Y_1^{\theta}} \leqslant \big(M^{(p_0)}(Y_0)\big)^{1-\theta} \big(M^{(p_1)}(Y_1)\big)^{\theta} K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta},$$

where

$$\log K_\theta(s) = \int_{\mathbb{R}} P_0(\theta, t) \log K_0(t+s)\, dt + \int_{\mathbb{R}} P_1(\theta, t) \log K_1(t+s)\, dt.$$

Lemma Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure space (Ω, Σ, μ) such that X_0 is p_0-convex and X_1 is p_1-convex. Then for every $0 < \theta < 1$ we have

$$\|x\|_{X_0^{1-\theta} X_1^{\theta}} \leqslant (M^{(p_0)}(X_0))^{1-\theta} (M^{(p_1)}(X_1))^{\theta} \, \|x\|_{(X_0, X_1)_\theta}, \quad x \in X_0 \cap X_1.$$

In particular (X_0, X_1) is an admissible quasi-Banach couple.

Lemma Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure (Ω, Σ, μ). If $x_j \in X_j$ are such that $|x_j|$ $(j = 0, 1)$ are bounded above and their non-zero values have positive lower bounds, then

$$|x_0|^{1-\theta} |x_1|^{\theta} \in (X_0, X_1)_\theta$$

and

$$\big\| |x_0|^{1-\theta} |x_1|^{\theta} \big\|_{(X_0, X_1)_\theta} \leqslant \|x_0\|_{X_0}^{1-\theta} \|x_1\|_{X_1}^{\theta}.$$

Corollary Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure space (Ω, Σ, μ). If $x \in X_0 \cap X_1$ has an order continuous norm in $X_0^{1-\theta} X_1^{\theta}$, then for every $0 < \theta < 1$,

$$\|x\|_{(X_0, X_1)_\theta} \leqslant \|x\|_{X_0^{1-\theta} X_1^{\theta}}.$$

Theorem Let (X_0, X_1) be a couple of complex quasi-Banach lattices on a measure space with nontrivial lattice convexity constants. If the space $X_0^{1-\theta} X_1^{\theta}$ has order continuous quasi-norm, then

$$[X_0, X_1]_\theta = X_0^{1-\theta} X_1^{\theta}$$

up equivalences of norms (isometrically, provided that lattice convexity constants are equal to 1). In particular this holds if at least one of the spaces X_0 or X_1 is order continuous.

Theorem For each $1 \leqslant i \leqslant m$, let (X_{0i}, X_{1i}) be complex quasi-Banach function lattices and let Y_j be complex p_j-convex maximal quasi-Banach function lattices with p_j-convexity constants equal 1 for $j = 0, 1$. Suppose that either X_{0i} or X_{1i} is order continuous for each $1 \leqslant i \leqslant m$. Let T be a multilinear operator defined on $(X_{01} + X_{11}) \times \cdots \times (X_{0m} + X_{1m})$ and taking values in $Y_0 + Y_1$ such that

$$T\colon X_{i1} \times \cdots \times X_{im} \to Y_i$$

is bounded with quasi-norm M_i for $i = 0, 1$. Then for $0 < \theta < 1$,

$$T\colon (X_{01})^{1-\theta}(X_{11})^{\theta} \times \cdots \times (X_{0m})^{1-\theta}(X_{1m})^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}$$

is bounded with the quasi-norm

$$\|T\| \leqslant M_0^{1-\theta} M_1^{\theta}.$$

As an application we obtain the following interpolation theorem for operators was proved by Kalton (1990), which was applied to study a problem in uniqueness of structure in quasi-Banach lattices (Kalton's proof uses a deep theorem by Nikishin and the theory of Hardy H_p-spaces on the unit disc).

Theorem Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measures spaces. Let X_i, $i = 0, 1$, be complex p_i-convex quasi-Banach lattices on $(\Omega_1, \Sigma_1, \mu_1)$ and let Y_i be complex p_i-convex maximal quasi-Banach lattices on $(\Omega_2, \Sigma_2, \mu_2)$ with p_i-convexity constants equal 1. Suppose that either X_0 or X_1 is order continuous. Let $T\colon X_0 + X_1 \to L^0(\mu_2)$ be a continuous operator such that $T(X_0) \subset Y_0$ and $T(X_1) \subset Y_1$. Then for $0 < \theta < 1$,

$$T\colon X_0^{1-\theta} X_1^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}$$

and

$$\|T\|_{X_0^{1-\theta} X_1^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}} \leqslant \|T\|_{X_0 \to Y_0}^{1-\theta} \|T\|_{X_1 \to Y_1}^{\theta}.$$

The Lorentz space $L_{p,q} := L_{p,q}(\Omega)$ on a measure space (Ω, Σ, μ), $0 < p < \infty, 0 < q \leqslant \infty$ consists of all $f \in L^0(\mu)$ such that

$$\|f\|_{L_{p,q}} := \begin{cases} \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t} \right)^{1/q} & \text{if } 0 < q < \infty \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty\,. \end{cases}$$

- The decreasing rearrangement f^* of f with respect to μ is defined by

$$f^*(t) = \inf\{s > 0;\, \mu_f(s) \leqslant t\},\; t \geqslant 0,$$

where

$$\mu_f(s) = \mu(\{\omega \in \Omega;\, |f(\omega)| > s\}), \quad s > 0.$$

Lemma Let $0 < p_j, q_j < \infty$ and let L_{p_j,q_j} for $j = 0, 1$ be Lorentz spaces on an infinite nonatomic measure space (Ω, Σ, μ). Then for $0 < \theta < 1$ the quasi-norm of

$$X_\theta := (L_{p_0,q_0})^{1-\theta}(L_{p_1,q_1})^\theta$$

is equivalent to that of $L_{p,q}$, where $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Moreover for all $f \in X_\theta$ we have

$$2^{-1/p}\,\|f\|_{L_{p,q}} \leqslant \|f\|_{X_\theta} \leqslant \frac{2^{1/p}}{(\log 2)^s}\, s^s (p_0^{1-\theta} p_1^\theta)^s\, \|f\|_{L_{p,q}},$$

where $s = 1$ whenever $1 < p_0, p_1 < \infty$ and $1 \leqslant q_0, q_1 < \infty$ and $s > \max\{1/p_0, 1/q_0, 1/p_1, 1/q_1\}$ otherwise.

Theorem Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces. For $1 \leqslant i \leqslant m$, fix $0 < q_0, q_1, q_{0i}, q_{1i} < \infty$, $0 < r_0, r_1, r_{0i}, r_{1i} \leqslant \infty$ and for $0 < \theta < 1$, define q, r, q_i, r_i by setting

$$\frac{1}{q_i} = \frac{1-\theta}{q_{0i}} + \frac{\theta}{q_{1i}}, \quad \frac{1}{r_i} = \frac{1-\theta}{r_{0i}} + \frac{\theta}{r_{1i}}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

Assume that $\mathcal{X}_i$ is a dense linear subspace of $L_{q_{0i},r_{0i}}(\Omega_1) \cap L_{q_{1i},r_{1i}}(\Omega_1)$ and that $\{T_z\}_{z\in\overline{S}}$ is an admissible analytic family of multilinear operators $T_z\colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to L_{q_0,r_0}(\Omega_2) \cap L_{q_1,r_1}(\Omega_2)$. Suppose that for every $(h_1, \dots, h_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $t \in \mathbb{R}$ and $j = 0, 1$, we have

$$\|T_{j+it}(h_1, \dots, h_m)\|_{L_{q_j,r_j}(\Omega_2)} \leqslant K_j(t)\|h_1\|_{L_{q_{j1},r_{j1}}(\Omega_1)} \cdots \|h_m\|_{L_{q_{jm},r_{jm}}(\Omega_1)}$$

where K_j are Lebesgue measurable functions such that $K_j \in L^{p_j}(P_j(\theta,\cdot)\,dt)$ for all $\theta \in (0,1)$, where p_j is chosen so that $0 < p_j < q_j$ and $p_j \leqslant r_j$ for each $j = 0, 1$.

Then for all $(f_1, \ldots, f_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $0 < \theta < 1$, and $s \in \mathbb{R}$ we have

$$\| T_{\theta+is}(f_1, \ldots, f_m)\|_{(L_{q_0,r_0}, L_{q_1,r_1})_\theta} \leqslant C_\theta \, K_\theta(s) \prod_{i=1}^{m} \|f_i\|_{(L_{q_0,r_0}, L_{q_1,r_1})_\theta},$$

where

$$C(\theta) := \Big(\frac{q_0}{q_0 - p_0}\Big)^{\frac{1-\theta}{p_0}} \Big(\frac{q_1}{q_1 - p_1}\Big)^{\frac{\theta}{p_1}},$$

$$\log K_\theta(s) = \int_{\mathbb{R}} P_0(\theta, t) \log K_0(t+s)\, dt + \int_{\mathbb{R}} P_1(\theta, t) \log K_1(t+s)\, dt.$$

If in addition the measures spaces are infinite and nonatomic, then for all $(f_1, ..., f_m)$ in $\mathcal{X}_1 \times \cdots \times \mathcal{X}_m$ and $s \in \mathbb{R}$, and $0 < \theta < 1$ we have

$$\|T_{\theta+is}(f_1, ..., f_m)\|_{L_{q,r}(\Omega_2)} \leqslant C \left(\frac{q_0}{q_0 - p_0}\right)^{\frac{1-\theta}{p_0}} \left(\frac{q_1}{q_1 - p_1}\right)^{\frac{\theta}{p_1}} K_\theta(s) \prod_{i=1}^{m} \|f_i\|_{L_{q_i,r_i}(\Omega_1)},$$

where

$$C = 2^{\frac{1}{q}+\sum_{i=1}^{m}\frac{1}{q_i}} \left(\frac{u\, q_0^{1-\theta} q_1^{\theta}}{\log 2}\right)^u$$

with $u = 1$ if $1 < q_0, q_1 < \infty$ and $1 \leqslant r_0, r_1 \leqslant \infty$, while $u > \max\{1/q_0, 1/q_1, 1/r_0, 1/r_1\}$ otherwise.

Suppose that there is an operator $\mathcal{M}$ defined on a linear subspace of $\widetilde{L}^0(\Omega, \Sigma, \mu)$ and taking values in $\widetilde{L}^0(\Omega, \Sigma, \mu)$ such that:

(a) For $j = 0$ and $j = 1$ the function $(t, x) \mapsto \mathcal{M}(h(j + it, \cdot))(\omega)$, $(t, \omega) \in \mathbb{R} \times \Omega$ is $\mathcal{L} \times \Sigma$-measurable for any function $h \colon \partial S \times \Omega \to \mathbb{C}$ such that $\omega \mapsto h(j + it, \omega)$ is Σ-measurable for almost all $t \in \mathbb{R}$.

(b) $\mathcal{M}(\lambda h)(\omega) = |\lambda| \mathcal{M}(h)(\omega)$ for all $\lambda \in \mathbb{C}$.

(c) For every function h as in above there is an exceptional set $E_h \in \Sigma$ with $\mu(E_h) = 0$ such that for $j \in \{0, 1\}$

$$\mathcal{M}\left(\int_{-\infty}^{\infty} h(t, \cdot) P_j(\theta, t)\, dt \right)(\omega) \leqslant \int_{-\infty}^{\infty} \mathcal{M}(h(t, \cdot))(\omega) P_j(\theta, t)\, dt$$

for all $z \in \mathbb{C}$, all $\theta \in (0, 1)$, and all $\omega \notin E_h$. Moreover, $E_{\psi h} = E_h$ for every analytic function ψ on S which is bounded on $\overline{S}$.

For each $1 \leqslant i \leqslant m$, let $\overline{X}_i = (X_{0i}, X_{1i})$ be admissible couples of quasi-Banach spaces, and let (Y_0, Y_1) be a couple of complex maximal quasi-Banach lattices on a measure space (Ω, Σ, μ) such that each Y_j is p_j-convex for $j = 0, 1$. Assume that $\mathcal{X}_i$ is a dense linear subspace of $X_{0i} \cap X_{1i}$ for each $1 \leqslant i \leqslant m$, and that $\{T_z\}_{z \in \overline{S}}$ is an admissible analytic family of multilinear operators $T_z \colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to Y_0 \cap Y_1$. Assume that $\mathcal{M}$ is defined on the range of T_z, takes values in $L^0(\Omega, \Sigma, \mu)$, and satisfies conditions (a), (b) and (c).

Theorem Suppose that for every $(x_1, ..., x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $t \in \mathbb{R}$ and

$$\|\mathcal{M}(T_{j+it}(x_1, ..., x_m))\|_{Y_j} \leqslant K_j(t)\|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}, \quad j = 0, 1,$$

where K_j are Lebesgue measurable functions such that $K_j \in L^{p_j}(P_j(\theta, \cdot)\, dt)$ for all $\theta \in (0, 1)$. Then for all $(x_1, ..., x_m) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_m$, $s \in \mathbb{R}$, and $0 < \theta < 1$,

$$\|\mathcal{M}(T_{\theta+is}(x_1, ..., x_m))\|_{Y_0^{1-\theta} Y_1^{\theta}} \leqslant C_\theta\, K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta},$$

where

$$C_\theta = \left(M^{(p_0)}(Y_0)\right)^{1-\theta} \left(M^{(p_1)}(Y_1)\right)^{\theta},$$

$$\log K_\theta(s) = \int_{\mathbb{R}} P_0(\theta, t) \log K_0(t + s)\, dt + \int_{\mathbb{R}} P_1(\theta, t) \log K_1(t + s)\, dt.$$

The preceding theorem has an important application to interpolation of multilinear operators that take values in Hardy spaces. A particular case of the above Theorem arises when:

- $\Omega = \mathbb{R}^n$, μ is Lebesgue measure, and

$$\mathcal{M}(h)(x) = \sup_{\delta>0} |\phi_\delta * h(x)|, \quad x \in \mathbb{R}^n$$

where ϕ is a Schwartz function on $\mathbb{R}^n$ with nonvanishing integral.

- $Y_0 = L^{p_0}$, $Y_1 = L^{p_1}$, in which case $Y_0^{1-\theta} Y_1^\theta = L^p$, where $1/p = (1-\theta)/p_0 + \theta/p_1$.

Definition The classical Hardy space H^p of Fefferman and Stein is defined by

$$\|h\|_{H^p} := \|\mathcal{M}(h)\|_{L^p}.$$

Corollary If $\{T_z\}$ is an admissible analytic family is such that

$$\|T_{j+it}(x_1, \dots, x_m)\|_{H^{p_j}} \leqslant K_j(t)\|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}, \quad j = 0, 1,$$

then

$$\|T_{\theta+s}(x_1, \dots, x_m)\|_{H^p} \leqslant K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta}$$

for $0 < p_0, p_1 < \infty$, $s \in \mathbb{R}$, and $0 < \theta < 1$. Analogous estimates hold for the Hardy-Lorentz spaces $H^{q,r}$ where estimates of the form

$$\|T_{j+it}(x_1, \dots, x_m)\|_{H^{q_j, r_j}} \leqslant K_j(t)\|x_1\|_{X_{j1}} \cdots \|x_m\|_{X_{jm}}$$

for admissible analytic families $\{T_z\}$ when $j = 0, 1$ imply

$$\|T_{\theta+is}(x_1, \dots, x_m)\|_{H^{q,r}} \leqslant C\, K_\theta(s) \prod_{i=1}^{m} \|x_i\|_{(X_{0i}, X_{1i})_\theta},$$

where

$$C := 2^{\frac{1}{q}} \left(\frac{u\, q_0^{1-\theta} q_1^{\theta}}{\log 2}\right)^{u} \left(\frac{q_0}{q_0 - p_0}\right)^{\frac{1-\theta}{p_0}} \left(\frac{q_1}{q_1 - p_1}\right)^{\frac{\theta}{p_1}},$$

$0 < p_j < q_j < \infty$, $p_j \leqslant r_j \leqslant \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$ while $u = 1$ if $1 < q_0, q_1 < \infty$ and $1 \leqslant r_0, r_1 \leqslant \infty$

Stein's motivation to study analytic families of operators might have been the study of the Bochner-Riesz operators

$$B^{\delta}(f)(x) := \int_{|\xi|\leqslant 1} \left(1-|\xi|^2\right)^{\delta}\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi.$$

in which the "smoothness" variable δ affects the degree p of integrability of $B^{\delta}(f)$ on $L^p(\mathbb{R}^n)$. Here f is a Schwartz function on $\mathbb{R}^n$ and $\widehat{f}$ is its Fourier transform defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx, \quad \xi \in \mathbb{R}^n.$$

Remark Using interpolation for analytic families of operators, Stein showed that whenever $\delta > (n-1)|1/p - 1/2|$, then

$$B^{\delta} \colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

is bounded for every $1 \leqslant p \leqslant \infty$.

- The bilinear Bochner-Riesz operators are defined on $\mathcal{S} \times \mathcal{S}$ by

$$S^{\delta}(f,g)(x) := \iint_{|\xi|^2+|\eta|^2 \leqslant 1} \left(1 - |\xi|^2 - |\eta|^2\right)^{\delta} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta$$

for every $f, g \in \mathcal{S}$.

- The bilinear Bochner-Riesz means S^z is defined by

$$S^z(f,g)(x) = \int\int K_z(x-y_1, x-y_2) f(y_1) g(y_2) dy_1\, dy_2,$$

where that the kernel of $S^{\delta+it}$ is given by

$$K_{\delta+it}(x_1,x_2) = \frac{\Gamma(\delta+1+it)}{\pi^{\delta+it}} \frac{J_{\delta+it+n}(2\pi|x|)}{|x|^{\delta+it+n}}, \quad x = (x_1,x_2).$$

If $\delta > n - 1/2$, then using known asymptotics for Bessel functions we have that this kernel satisfies an estimate of the form:

$$|K_{\delta+it}(x_1, x_2)| \leqslant \frac{C(n+\delta+it)}{(1+|x|)^{\delta+n+1/2}},$$

where $C(n+\delta+it)$ is a constant that satisfies

$$C(n+\delta+it) \leqslant C_{n+\delta} e^{B\,|t|^2}$$

for some $B > 0$ and so we have

$$|K_{\delta+it}(x_1, x_2)| \leqslant C_{n+\delta}\, e^{B|t|^2} \frac{1}{(1+|x_1|)^{n+\epsilon}} \frac{1}{(1+|x_2|)^{n+\epsilon}},$$

with $\epsilon = \frac{1}{2}(\delta - n - 1/2)$. It follows that the bilinear operator $S^{\delta+it}$ is bounded by a product of two linear operators, each of which has a good integrable kernel. It follows that

$$K^{\delta+it} \colon L^1 \times L^1 \to L^{1/2}$$

with constant $K_1(t) \leqslant C'_{n+\delta} e^{B|t|^2}$ whenever $\delta > n - 1/2$.

Theorem Let $1 < p < 2$. For any $\lambda > (2n-1)(1/p - 1/2)$

$$S^\lambda \colon L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \to L^{p/2}(\mathbb{R}^n) \quad \textit{is} \quad \textit{bounded}.$$

Proof. We apply the main Theorem with $X_{01} = X_{11} = L^2$, $X_{02} = X_{12} = L^1$, $Y_0 = L^1$, $Y_1 = L^{1/2}$, $\mathcal{X}_i$ is the space of Schwartz functions on $\mathbb{R}^n$, which is dense in L^1 and L^2. We fix $\delta > 0$ and we consider the bilinear analytic family $\{T_z\}_{z\in\overline{S}}$, where $T_z := S^{(n-\frac{1}{2})z+\delta}$ for all $z \in \overline{S}$. We claim that this family is admissible. Indeed, for f, g Schwartz functions we have

$$T_z(f,g)(x) = \iint_{|\xi|^2+|\eta|^2 \leqslant 1} \left(1 - |\xi|^2 - |\eta|^2\right)^{(n-\frac{1}{2})z+\delta} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta,$$

and the map $z \mapsto T_z(f,g)$ is analytic in S, continuous and bounded on $\overline{S}$, and jointly measurable in (t,x) when $z = it$ or $z = 1 + it$. Moreover, for all $x \in \mathbb{R}^n$ we have

$$\sup_{z\in\overline{S}} \frac{\log|T_z(f,g)(x)|}{e^{\alpha|\mathrm{Im}\, z|}} < \infty, \quad f, g \in \mathcal{S}$$

with $\alpha = 0 < \pi$; in fact $|T_z(f,g)(x)| \leqslant \|\widehat{f}\,\|_{L^1}\|\widehat{g}\,\|_{L^1}$.

Based on the preceding discussion, we have that when $\operatorname{Re} z = 0$, T_z maps $L^2 \times L^2$ to L^1 with constant $K_0(t) \leqslant C_{n,\delta} e^{c\,|t|^2}$ for some $C_{n,\delta}, c > 0$. We also have that when $\operatorname{Re} z = 1$, T_z maps $L^1 \times L^1$ to $L^{1/2}$ with constant $K_1(t) \leqslant C'_{n,\delta} e^{B\,|t|^2}$ for some $C'_{n,\delta}$, $B > 0$. We notice that for these functions $K_i(t)$ we have that the constant $K(\theta, 1, 1/2) < \infty$; in this case $\theta = 2(\frac{1}{p} - \frac{1}{2})$. An application of the main Theorem yields that

$$S^\lambda \colon L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \to L^{p/2}(\mathbb{R}^n)$$

provided $\lambda = 2(n - \frac{1}{2})\,(\frac{1}{p} - \frac{1}{2}) + \delta > (2n-1)(\frac{1}{p} - \frac{1}{2})$.