The Kowalski-Słodkowski theorem as a pioneering result in the theory of 2-local automorphisms

Antonio M. Peralta

Departamento de Análisis Matemático
Universidad de Granada

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Our story begins in 1967-1968. During these years A.M. Gleason [*J. d'Anal. Math.*'1967] and J.-P. Kahane and W. Żelazko [*Studia Math.*'1968] established, independently, a fundamental result in the theory of Banach algebras.

[The Gleason-Kahane-Zelazko theorem]

Let A be a (non necessarily unital nor commutative) complex Banach algebra, and let $F : A \to \mathbb{C}$ be a continuous, non-zero, linear functional. The following are equivalent:

- (a) F is unital if A contains a unit or there exits a unital extension of F to the unitization on A, and $F(a) \neq 0$ whenever a is invertible in A;
- (b) For each $a \in A$, $F(a)$ belongs to the spectrum, $\sigma(a)$, of a;
- (c) F is multiplicative, that is $F(ab) = F(a)F(b)$, for every $a, b \in A$.

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Lagrange's mean value theorem and the fundamental theorem of calculus assure that the mapping

$$F : C([0,1],\mathbb{R}) \to \mathbb{R},$$

$$F(f) := \int_0^1 f(t)dt$$

satisfies statement (b) in the Gleason-Kahane-Zelazko theorem because for each f there exists $t_f \in [0,1]$ satisfying $F(f) := \int_0^1 f(t)dt = f(t_f) \in \sigma(f)$. However F is not multiplicative.

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However F is not multiplicative.

The real setting is not appropriate for our business!

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A linear mapping T from a Banach algebra A into a Banach algebra B is said to be a *local homomorphism* if for every a in A there exists a homomorphism $\Phi_a : A \to B$, depending on a, satisfying $T(a) = \Phi_a(a)$.

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Given a Banach algebra A and a a Banach A-bimodule X. Let me recall that a derivation from A into X is a liner map $D : A \to X$ satisfying

$$D(ab) = D(a)b + aD(b),$$

for every $a, b \in A$.

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Let me offer a unifying approach which valid for all definitions:

Local (linear) maps

Let $\mathcal{S}$ be a subset of the space $L(X, Y)$ of all linear maps between Banach spaces X and Y. A linear map $T : X \to Y$ is called a *local $\mathcal{S}$ map* if for each $x \in X$ there exists $S_x \in \mathcal{S}$, depending on x, such that $T(x) = S_x(x)$.

For $\mathcal{S} = \text{Der}(A, X)$ the set of all linear derivations from a Banach algebra into a Banach A-bimodule, we get the notion of *local derivation*. For $\mathcal{S} = \text{Hom}(A, B)$ the set of all linear homomorphisms from A into another Banach algebra B, local $\text{Hom}(A, B)$ are called *local homomorphisms*.

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The challenge consists in determining certain properties of the set $\mathcal{S} \subseteq L(X, Y)$ to assure that every local $\mathcal{S}$ map lies in $\mathcal{S}$.

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Let me forewarn you that, in some extremal cases, the game finishes quickly. For $\mathcal{S} = \mathcal{F}(X, Y)$ the set all finite-rank operators between Banach spaces X and Y, **every** linear mapping $T : X \to Y$ is a local $\mathcal{S}$-map.

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Corollary [Gleason-Kahane-Zelazko, Studia'1968]

Every unital bounded linear local homomorphism from a unital complex Banach algebra A into $\mathbb{C}$ is multiplicative.

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We can easily deduce from Gelfand theory:

Corollary

Let $T : A \to B$ be a bounded local homomorphism between C*-algebras, where B is commutative. Then T is a Jordan *-homomorphism.

This is just the beginning of the story!!!

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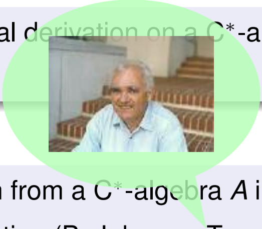

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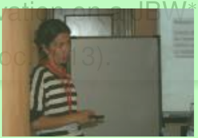

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Weak-local (linear) maps [Essaleh, Peralta, Ramirez, Lin. Mult. Algebra'2015]

Let $\mathcal{S}$ be a subset of the space $L(X, Y)$ of all linear maps between Banach spaces X and Y. A linear map $T : X \to Y$ is called a *weak-local $\mathcal{S}$ map* if for each $x \in X$ and every $\phi \in Y^*$, there exists $S_{x,\phi} \in \mathcal{S}$, depending on x and ϕ, such that $\phi T(x) = \phi S_{x,\phi}(x)$.

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We pay special attention to the cases in which $\mathcal{S}$ is the set of all derivations, or the set of all *-automorphisms on a C*-algebra A. These linear maps are called, weak-local derivations and weak-local *-automorphisms, respectively.

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Let X be a Banach A-bimodule on a Banach algebra A. We recall that a linear mapping $D : A \to X$ is a derivation if

$$D(ab) = D(a)b + aD(b),$$

for every $a, b \in A$.

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Every weak-local derivation on a C*-algebra A is continuous.

The proof is based on techniques developed for dissipative linear maps between Banach spaces applied by authors like Kadison, Ringrose and Kishimoto.

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Every weak-local derivation on a C*-algebra A is a derivation.

The next natural goal is to consider weak-local *-automorphisms. In this case we do not know the answer. However, under stronger hypothesis, we can throw some light into this problem.

Let us consider a more general notion:

Let X and Y be a Banach spaces, and let τ be a locally convex topology on Y defined by a family of seminorms $\left\{|||.|||_i : i \in I\right\}$. Let $\mathcal{S}$ be a subset in $L(X, Y)$. A mapping $T \in L(X, Y)$ will be called a τ-local $\mathcal{S}$ map if

$$\text{for each } x \in X \text{ and each } i \in I, \text{ there exists } T_{x,i} \in \mathcal{S}, \tag{1}$$

$$\text{depending on } x \text{ and } i, \text{ such that } |||T(x) - T_{x,i}(x)|||_i = 0.$$

We define in this way τ-local derivations and τ-local automorphisms among many other possible notions.

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Every strong-local *-automorphism on a von Neumann algebra is a Jordan *-homomorphism.

By a criterium established by Murnaghan in 1954, we know that every matrix in $M_2(\mathbb{C})$ is unitarily equivalent to its transpose, that is for each $a \in M_2(\mathbb{C})$ there exists a unitary matrix $u \in M_2(\mathbb{C})$ (depending on a) satisfying $u^* a u = a^t$. Consequently, the mapping

$$T : M_2(\mathbb{C}) \to M_2(\mathbb{C}), \quad T(a) = a^t,$$

is a linear local *-homomorphism and a *-anti-homomorphism which is not multiplicative. Therefore the above theorem is, in some sense, "optimal".

Algebra Strikes Back: The Kowalski-Słodkowski theorem (a 2-local behaviour)

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Theorem [Kowalski and Słodkowski, Studia'1980]

Suppose A is a complex Banach algebra (non necessarily commutative nor unital). Then every (non necessarily linear) mapping $T : A \to \mathbb{C}$ satisfying $T(0) = 0$ and $T(x) - T(y) \in \sigma(x - y)$, for every $x, y \in A$, is multiplicative and linear.

2-local maps

Let $\mathcal{S}$ be a subset of the space $L(X, Y)$ of all linear maps between Banach spaces X and Y. A (non necessarily linear nor continuous) mapping $\Delta : X \to Y$ is said to be a *2-local $\mathcal{S}$ map* if for every $x, y \in X$ there exists $T_{x,y} \in \mathcal{S}$, depending on x and y, such that

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2-local derivations, *2-local *-derivations*, *2-local homomorphisms*, *2-local *-homomorphisms, 2-local Jordan homomorphisms*, and *2-local automorphisms* between Banach algebras and C*-algebras are precisely 2-local $\mathcal{S}$ maps when $\mathcal{S}$ is the set of all derivations, *-derivations, homomorphisms, *-homomorphisms, Jordan homomorphisms, and automorphisms, respectively.

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$$T_{x,y}(x) = \Delta(x), \text{ and } T_{x,y}(y) = \Delta(y).$$

If we take $\mathcal{S} = K(X, Y)$ every 1-homogeneous map $\Delta : X \to Y$ is a 2-local $\mathcal{S}$-map.

Theorem [Kowalski and Słodkowski, Studia'1980]

Every (non necessarily linear) 2-local homomorphism from a (non necessarily commutative nor unital) complex Banach algebra A into the complex field $\mathbb{C}$ is linear and multiplicative. Consequently, every (non necessarily linear) 2-local homomorphism from A into a commutative C*-algebra is linear and multiplicative.

Another contributions.......

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Every 2-local derivation on a von Neumann algebra is a derivation.

Let A be a C*-algebra. A linear map $\delta : A \to A$ is said to be a triple derivation if

$$\delta\{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\},$$

for every $a, b, c \in A$, where $\{a, b, c\} = \frac{1}{2}(ab^* c + cb^* a)$.

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[Kudaybergenov, Oikhberg, Peralta, Russo, Illinois J. Math.'2014]

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for every $a,b,c \in A$, where $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$.

[Kudaybergenov, Oikhberg, Peralta, Russo, Illinois J. Math. 2014]

Every 2-local triple derivation on a von Neumann algebra is a triple derivation.

Linear 2-local maps

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[Peralta, Operators and Matrices'2015]

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Problem:

Let A and B be two C*-algebras and let $T : A \to B$ be a non necessarily linear nor continuous 2-local $*$-homomorphism. Is T linear and a $*$-homomorphism?

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Let $\mathcal{P}(M)$ denote the lattice of projections in a von Neumann algebra M, let X be a Banach space. A finitely additive, X-valued, bounded measure on $\mathcal{P}(M)$ is a mapping $\mu : \mathcal{P}(M) \to X$ such that

(*a*) $\mu(e+f) = \mu(e) + \mu(f)$ whenever $ef = 0$;

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Clearly each bounded linear operator from M to X restricts to a finitely additive X-valued measure.

Mackey-Gleason Problem:

Let M be a von Neumann algebra having no direct summand of Type I_2. Is every finitely additive X-valued bounded measure on M the restriction of a bounded linear operator from M into X?

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Let M be a von Neumann algebra with no direct summand of Type I_2. Then, for each Banach space X, each X-valued. finitely additive, bounded measure on $\mathcal{P}(M)$ has a unique extension to a bounded linear operator from M to X.

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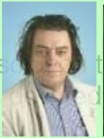

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$$T\left(\sum_{i=1}^{n} \lambda_i p_i\right) = \sum_{i=1}^{n} \lambda_i T(p_i),$$

for every set $\{p_1, \ldots, p_n\}$ of mutually orthogonal projections in M and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$.

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T is orthogonally additive, that is, $T(a+b) = T(a) + T(b)$ whenever $a \perp b$ in M.

Combining these technical properties with the Bunce-Wright-Mackey-Gleason theorem, we get:

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A new hope: weak-2-local symmetric maps

Let X and Y be a Banach spaces, and let τ be a locally convex topology on Y defined by a family of seminorms $\left\{ |||.|||_i : i \in I \right\}$. Let $\mathcal{S}$ be a subset in $B(X, Y)$. A mapping $\Delta : X \to Y$ will be called a τ-2-local $\mathcal{S}$ map if

$$\text{for each } x, y \in X \text{ and each } i \in I, \text{ there exists } T_{x,y,i} \in \mathcal{S}, \tag{2}$$

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Let A and B be C*-algebras. When $\mathcal{S}$ is the set of all linear maps $T : A \to B$ which are symmetric (i.e. $T(a^*) = T(a)^*$, for every $a \in A$), weak-2-local $\mathcal{S}$ maps are called weak-2-local symmetric maps.

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More on JB*-triples, Jordan theory and completely additive measures

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[Hamhalter, Kudaybergenov, Peralta, Russo, J. Math. Phys.'2016]

Every (non necessarily linear nor continuous) 2-local triple derivation on a continuous JBW*-triple is a triple derivation.