Some generalizations of Burnside's Theorem

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Outline

Notations

- All groups considered in this talk are finite;
- G always denotes a finite group;
- $\pi(G)$ denotes the set of all primes dividing the order of G;
- Suppose that p is a prime in $\pi(G)$. $O^p(G)$ is the subgroup of G generated by all p'-elements of G;
- $Z_k(G)$ is the k-th center of G, $k \geq 1$.
- In fact, $Z_1(G) = Z(G)$, the center of G, and $Z_k(G)/Z_{k-1}(G) = Z(G/Z_{k-1}(G))$ for $k > 1$.

Notations

If P is a p-group and k is a natural number, we denote

$$\Omega_k(P) = \langle x \in P : x^{p^k} = 1 \rangle,$$

$$\Omega(P) = \begin{cases} \Omega_1(P), & if \ \ \text{p} \ \ is \ \ odd; \\ \Omega_2(P), & if \ \ p = 2. \end{cases}$$

Notations

- *A* group G is called p-nilpotent if G has a normal p-complement;
- i.e., p does not divide the order of $O^p(G)$;
- i.e., $P \cap O^p(G) = 1$ for any Sylow p-subgroup P of G.

Notations

O. H. Kegel, Math. Z., 1962

A subgroup H of G is said to be *s-permutable (or s-quasinormal, π-quasinormal)* in G if H permutes with every Sylow subgroup of G.

Notations

Let L/K be a chief factor of G and H a subgroup of G. We say that

- i) *H covers L/K* if $L \leq HK$, i.e., $L/K \leq HK/K$;
- ii) *H avoids L/K* if $L \cap H \leq K$, i.e., $L/K \cap HK/K = 1$;
- iii) H has *the cover-avoidance property* in G, H is *a CAP-subgroup* of G in short, if H either covers or avoids every chief factor of G.

The cover-avoidance property of subgroup was first studied by Gaschütz in 1962 to study the solvable groups, later by many other experts.

Notations

Y. Fan, X. Guo and K. P. Shum, ***Chinese Ann. Math.*****, 2006**

Let H be a subgroup of a group G. Then, we say that H is a partial CAP-subgroup (or semi CAP-subgroup or SCAP-subgroup for some authors) of G if there exists a chief series Γ_H of G such that H either covers or avoids each chief factor of Γ_H.

Background

For a group G, we know that the normalizers of its Sylow subgroups give a lot of messages of the whole group G.

M. Bianchi, A. Gillio Berta Mauri and P. Hauck, Arch. Math., 1986

A group is nilpotent if and only if the normalizer of its each Sylow subgroup is nilpotent.

Background

Ballester-Bolinches and Shemetkov strengthen the above result to get:

Ballester-Bolinches and Shemetkov, Siberian Math. J., 1999

A group is nilpotent if and only if the normalizer of its each Sylow p-subgroup is p-nilpotent for any prime $p \in \pi(G)$.

Background

Here we consider the local version, i.e., referring one prime.

Now we assume that p is a fixed prime in $\pi(G)$ and P is a Sylow p-subgroup of G.

Background

The property of $N_G(P)$ still influences the structure of G.

Thompson *Suppose that* $p \geq 5$*,* P *is a Sylow* p*-subgroup of* G *and* $P \neq 1$*. If* $N_G(P)$ *is* p*-nilpotent, then* $O^p(G) < G$*.*

Background

Applying Thompson's result, it is not difficult to prove a long-standing conjecture of Zassenhaus.

Theorem *If G is a finite group and* $N_G(Q) = Q$ *for every Sylow subgroup Q of G, then* $|G|$ *is a power of a prime.*

Background

We can see in general:

$N_G(P)$ is p-nilpotent $\not\Rightarrow$ G is p-nilpotent.

Background

Example *Suppose that* $G = GL(2,3)$ *and* $p = 2$ *and* P *is a Sylow* 2*-subgroup of* G*. Then* $N_G(P) = P$ *is* 2*-nilpotent but* G *is not* 2*-nilpotent.*

Background

Hence, there is the following question:

Question *Suppose that $N_G(P)$ is p-nilpotent. Which extra condition can guarantee that G is p-nilpotent ?*

Background

In the literature, many experts have considered this question, for example, Burnside, P. Hall, Wielandt, Glauberman, Thompson, etc.

Glauberman-Thompson Let p be an odd prime divisor of the order of a group G and let P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(Z(J(P)))$ is p-nilpotent, where $J(P)$ is the Thompson subgroup of P.

Remark We note that $N_G(P) \subseteq N_G(Z(J(P)))$.

Background

Our works are from the most famous Burnside Theorem:

Burnside Theorem *Let p be a fixed prime in $\pi(G)$ and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if P is abelian.*

equivalent form:

Let p be a fixed prime in $\pi(G)$ and P a Sylow p-subgroup of G. Then G is p-nilpotent if $N_G(P) = C_G(P)$.

Generalization I of Burnside Theorem

- P is abelian $\Leftrightarrow P \leq Z(P)$. From this point of view,
- to extend Burnside Theorem, we must weaken the condition " $P \leq Z(P)$".
- We have two ways, one is to enlarge "$Z(P)$", another is to lessen " P".

Generalization I of Burnside Theorem

P. Hall, Proc. London Math. Soc., 1936

Let p be a fixed prime in $\pi(G)$ *and P a Sylow p-subgroup of G. Suppose that* $N_G(P)$ *is p-nilpotent. Then G is p-nilpotent if the nilpotency class of P is less than p.*

Remark the nilpotency class of P is less than p means that $P \leq Z_{p-1}(P)$

Generalization I of Burnside Theorem

G. Zhang, Proc. Amer. Math. Soc., 1986

Let G be a finite group and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if $\Omega_1(P) \leq Z(P)$ and $C_G(Z(P))$ is p-nilpotent.

Generalization I of Burnside Theorem

J. González-Sánchez and T. S. Weigel, Israel J. Math., 2011

Suppose that p is a prime. Let G be a group and P a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if one of the following holds:

- *1. $\Omega(P) \leq Z_{p-1}(P)$.*
- *2. when $p = 2$, $\Omega_1(P) \leq Z(P)$ and P is quaternion-free.*

Generalization I of Burnside Theorem

Remark J González-Sánchez and T. S. Weigel apply cohomology ring theory to obtain the result. Ballester-Bolinches, etc. give another approach based on the classical theory of Hall and Higman (see B. Huppert and N. Blackburn, Finite Groups II, Chapter IX).

Generalization I of Burnside Theorem

We mention the following result:

A. Ballester-Bolinches and X. Guo, J. Algebra, 2000

Let p be a prime dividing the order of a group G and let P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if one of the following holds:

- *1. $\Omega(P \cap G') \leq Z(P)$;*
- *2. when $p = 2$, $\Omega_1(P \cap G') \leq Z(P)$ and P is quaternion-free.*

where G' is the commutator subgroup of G.

Generalization I of Burnside Theorem

Remark

- $P \cap O^p(G) \leq P \cap G'$ if P is a Sylow p-subgroup of G;
- G is p-nilpotent if and only if $P \cap O^p(G) = 1$.

Generalization I of Burnside Theorem

Li, Su, Wang, Xie's Theorem 1

Suppose that p is a prime. Let G be a group and P a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if and only if one of the following holds:

- *1.* $\Omega(P \cap O^p(G)) \leq Z_{p-1}(P)$.
- *2. when* $p = 2$, $\Omega_1(P \cap O^p(G)) \leq Z(P)$ *and P is quaternion-free.*

Generalization I of Burnside Theorem

The works in the above is to weaken the condition " $P \leq Z(P)$":

- 1. enlarge $Z(P)$: $Z(P) \rightarrow Z_{p-1}(P)$;
- 2. lessen P: $P \rightarrow \Omega(P) \rightarrow \Omega(P \cap G') \rightarrow \Omega(P \cap O^p(G)) \rightarrow \Omega(P \cap G^{\mathcal{N}}) \rightarrow \Omega(P \cap G^{\mathcal{L}})$.

Where $G^{\mathcal{N}}$ is the nilpotent residual of G, $G^{\mathcal{L}}$ is the $\mathcal{L}$-residual of G, $\mathcal{L}$ is the class of all p-solvable groups whose p-lengths are at most 1.

Generalization II of Burnside Theorem

From the other view to extend Burnside's Theorem. We first mention two results in this line, one belongs to Wielandt, the other belongs to Ballester-Bolinches and Esteban-Romero.

Generalization II of Burnside Theorem

A p-group G is called *regular* if, for any $x, y \in G$, there holds the following:

$$x^p y^p = (xy)^p \prod d_i^p,$$

where $d_i \in \langle x, y \rangle'$.

Generalization II of Burnside Theorem

A p-group is regular if its nilpotency class is less than p. Hence we can see that Wielandt's following result is a generalization of Hall's , of course, Burnside's Theorem.

Wielandt, 1960, J. Amer Math. Soc.

Let p be a fixed prime in $\pi(G)$ and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if P is regular.

Generalization II of Burnside Theorem

A group G is called *modular* if, for any subgroups H and K of G, HK is a subgroup of G.

Generalization II of Burnside Theorem

A. Ballester-Bolinches and R. Esteban-Romero, J. Algebra, 2002

Let p be a fixed prime in $\pi(G)$ and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if P is modular.

Generalization II of Burnside Theorem

Hall's result and the above result also follow from Yoshida¡¯s transfer theorem:

Yoshida's theorem

The normalizer of a Sylow p-subgroup P of a group G controls p-transfer unless P has a homomorphic image isomorphic to $C_p \wr C_p$, where $\wr$ is the wreath product.

Generalization II of Burnside Theorem

Questions

- 1. Does the theorem hold if P is other kind of p-group ? For example, meta-cyclic, meta-abelian, p-group of maximal class, Powerful p-group, etc. **the answer is no ! see Example 2 in the last section.**
- 2. Suppose that P is meta-cyclic. If $p > 2$, a meta-cyclic p-group is p-regular, hence the answer is yes; but in the case $p = 2$, **the answer is no !** S_4 **is a counter-example.**
- 3. Does the theorem hold if $P \cap O^p(G)$ is a special kind of p-group ? **The answer is no ! see Example 1 in the last section.**

Generalization II of Burnside Theorem

I think there exists a generalized regular property of p-group or a generalized modular property of p-group such that, under the assumption that $N_G(P)$ is p-nilpotent, G is p-nilpotent if and only if P possesses this property.

Generalization III of Burnside Theorem

" P is abelain " $\Leftrightarrow$ " $P' = 1$ "

From this point of view, we have:

Li, Su, Wang, Xie's Theorem 2

Let p be a fixed prime in $\pi(G)$ and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if P' is normal in G.

Generalization III of Burnside Theorem

Go further:

Li, Su, Wang, Xie's Theorem 3
Let p be a fixed prime in $\pi(G)$ and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if P' is s-permutable in G.

Corollary (X. Guo and X. Zhao, Acta Mathimatica Scientia, 2008)
Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every maximal subgroup of P is s-permutable in $N_G(P)$ and P' is s-permutable in G, then G is p-nilpotent.

Generalization III of Burnside Theorem

Known results:

- Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every maximal subgroup of P is s-permutable in G, then G is p-nilpotent.
- (Schmid, J Algbera, 1998) if H is a normal subgroup of a Sylow subgroup of a group G and H is s-permutable in G, then H is normal in G.

Generalization III of Burnside Theorem

Similarly, we consider other generalized normalities.

Li, Su, Wang, Xie's Theorem 4

Let p be a fixed prime in $\pi(G)$ and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if and only if P' is a partial CAP-subgroup of G.

Generalization III of Burnside Theorem

Known result:
G is p-nilpotent if every maximal subgroup of P is a CAP-subgroup of G, where p is the smallest prime in $\pi(G)$ and P is a Sylow p-subgroup of G.

Corollary (X. Guo, ect, Publ. Math. Debrecen, 2011)
Let p be the smallest prime in $\pi(G)$ and P a Sylow p-subgroup of G. Then G is p-nilpotent if every maximal subgroup of P is a CAP-subgroup of $N_G(P)$ and P' is a CAP-subgroup of G.

Generalization III of Burnside Theorem

Naturally, we have the following questions:

- 1. Can we replace the subgroup P' by other kinds of subgroups of P in the above theorems ? For example, $\Phi(P)$, etc., .
- 2. Can we unify the above theorems.

Generalization III of Burnside Theorem

We have the following result which unify the above theorem and Hall's result:

Li, Su, Wang and Xie's Theorem 5

Let p be a fixed prime in $\pi(G)$ and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if and only if there exists a normal subgroup N of P contained in $\Phi(P)$ such that N is a partial CAP-subgroup of G and the nilpotency class of P/N is less than p, i.e., $P/N \leq Z_{p-1}(P/N)$.

Generalization III of Burnside Theorem

Remark

- 1. Any normal subgroup N of P with $P' \leq N \leq \Phi(P)$ satisfies the conditions in the above theorem: the nilpotency class of P/N is less than p. Hence this condition is not much demanding.
- 2. If G is p-nilpotent, then every p-subgroup of G is a CAP subgroup of G. So the condition that N is a partial CAP-subgroup of G is natural.
- 3. If $P \leq Z_{p-1}(P)$, then pick $N = 1$. We obtain Hall's theorem; If P' is a CAP-subgroup of G, P/P' is abelian. Hence we obtain our Theorem 4.

Generalization III of Burnside Theorem

We also have some conjectures:

Conjecture 1 *Let p be a fixed prime in* $\pi(G)$ *and P a Sylow p-subgroup of G. Suppose that* $N_G(P)$ *is p-nilpotent. Then G is p-nilpotent if and only if there exists a normal subgroup N of P contained in* $\Phi(P)$ *such that N is a partial CAP-subgroup of G and* P/N *satisfies the following:*
1. $\Omega(P/N) \leq Z_{p-1}(P/N)$.
2. When $p = 2$, $\Omega_1(P/N) \leq Z(P/N)$ *and* P/N *is quaternion-free.*

Generalization III of Burnside Theorem

Conjecture 2 *Let p be a fixed prime in $\pi(G)$ and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if and only if there exists a normal subgroup N of P such that $N \leq \Phi(P)$, P/N is a modular p-group and N is a partial CAP-subgroup of G.*

Final Remark

We know that G is p-nilpotent iff $P \cap O^p(G) = 1$.
The following example shows that G may not be p-nilpotent if we assume that $P \cap O^p(G)$ is abelian or normal in G, under the hypothesis that $N_G(P)$ is p-nilpotent.

Final Remark

Example 1

Let $H = A_4 = C_3[B_4]$(the alternating group on 4 symbols).
This has a faithful irreducible module of dimension 3 over the field of 3 elements $GF(3)$. Call this module $N = C_3 \times C_3 \times C_3$.
Let $G = [N]H = [N](C_3[B_4])$.
Assume that $p = 3$.
Then $G_3 = P = [N]C_3$, $O^3(G) = NB_4$.
$P \cap O^3(G) = N$ is abeian and normal in G.
But G is not 3-nilpotent.

Final Remark

Example 2 Suppose that $G = S_4$, the symmetric group of degree 4, $p = 2$, P is a Sylow 2-group, then $N_G(P) = P = D_8$, D_8 is meta-cyclic, but G is not 2-nilpotent.

Final Remark

- Thanks for all presenting here !