Sequence spaces of type S^ν, and beyond with wavelet leaders

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Context

- Study of the pointwise regularity of a signal f by means of the Hölder exponents h_f

$$h_f(x_0) = \sup\{\alpha \geq 0 \ : \ f \in C^\alpha(x_0)\}$$

and multifractal formalisms, which are formulas that are expected to yield the spectrum of singularities of f defined by

$$d_f(h) = \dim_{\mathcal{H}}\{x \in \mathbb{R} \ : \ h_f(x) = h\}.$$

- Signals of $L^2([0,1])$ are represented through their wavelet coefficients

$$f = \sum_{j \geq 0} \sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}$$

and the Hölder regularity can be characterized using this representation.

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- The classical use of Besov spaces leads to a loss of information (for example, only the concave hull and the increasing parts of the spectrum can be recovered).
- S. Jaffard introduced spaces of type S^ν

  Aim : Detection of non concave spectrum.
- More recently, introduction of spaces of the same type but based on the wavelet leaders of the signal

  Aim : Detection of non increasing spectrum and of the oscillating singularities.

Let c be the sequence of wavelet coefficients of f. The wavelet profile ν_f of f is defined by

$$\nu_f(\alpha) = \lim_{\varepsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\log \# E_j(1, \alpha + \varepsilon)(f)}{\log 2^j} \right) \right)$$

for all $\alpha \in \mathbb{R}$, where

$$E_j(C, \alpha)(f) = \left\{ k : |c_{j,k}| \geq C 2^{-\alpha j} \right\}.$$

- **Interpretation**: there are "approximatively" $2^{\nu_f(\alpha) j}$ coefficients greater in modulus than $2^{-\alpha j}$.
- ν_f is independant of the chosen wavelet basis.

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Consider an increasing function $\nu : \mathbb{R} \to \{-\infty\} \cup [0,1]$ right continuous (called an admissible profile). Define

$$\begin{aligned} \alpha_{min} &:= \inf\{\alpha : \nu(\alpha) \geq 0\} \\ \alpha_{max} &:= \inf\{\alpha : \nu(\alpha) = 1\}. \end{aligned}$$

Denote Ω the set of complex sequences

$$c = (c_{j,k})_{j\in\mathbb{N},k\in\{0,\dots,2^j-1\}}.$$

Définition

The space $\mathbf{S}^\nu$ is the set of sequences $c \in \Omega$ such that

$$\forall \alpha \in \mathbb{R}, \forall \varepsilon > 0, \forall C > 0, \exists J \geq 0 : \#E_j(C,\alpha)(c) \leq 2^{(\nu(\alpha)+\varepsilon)j}, \forall j \geq J$$

where

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Proposition

The space S^ν is a vector space and

$$\mathrm{S}^\nu = \{c \in \Omega : \nu_c(\alpha) \leq \nu(\alpha) \, , \forall \alpha \in \mathbb{R}\} \, .$$

Examples of S^ν spaces :

- Assume that $\nu(\alpha) = 1$ for all $\alpha \in \mathbb{R}$. If $c \in \Omega$, for every $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$,

$$\# E_j(C, \alpha)(c) \leq 2^j < 2^{(\nu(\alpha)+\varepsilon)j} \quad \forall j \in \mathbb{N} \, .$$

  This means that $c \in \mathrm{S}^\nu$ and therefore, $\mathrm{S}^\nu = \Omega$.

- Assume that

$$\nu(\alpha) = \begin{cases} -\infty & \text{si} \quad \alpha < a \\ 1 & \text{si} \quad \alpha \geq a \end{cases}$$

  where $a \in \mathbb{R}$. Then S^ν is the set of sequences c such that for every $\alpha < a$,

$$\sup_{j \in \mathbb{N}} \sup_{k \in \{0, \ldots, 2^j - 1\}} 2^{\alpha j} |c_{j,k}| < +\infty.$$

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Besov Spaces

For $s \in \mathbb{R}$ and $p > 0$, a function belongs to the Besov space $b^s_{p,\infty}$ if its wavelets coefficients satisfy

$$\|c\|_{b^s_{p,\infty}} := \sup_{j\in\mathbb{N}_0} 2^{(s-\frac{1}{p})j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^p \right)^{\frac{1}{p}} < +\infty.$$

The definition is extended to the case $p = \infty$ by setting

$$\|c\|_{C^s} := \sup_{j\in\mathbb{N}_0} \sup_{k\in\{0,\dots,2^j-1\}} 2^{sj} |c_{j,k}|.$$

This corresponds to the Hölder space of order s, denoted by C^s. These spaces are independent of the wavelet mother chosen. Considered as sequence spaces, they are Banach spaces if $p \geq 1$ and complete metric spaces if $p < 1$.

If we define the concave conjugate η of ν by

$$\eta(p) := \inf_{\alpha \geq \alpha_{\min}} (\alpha p - \nu(\alpha) + 1)$$

we get the following characterization of S^ν spaces.

Link with Besov Spaces

If $(p_n)_{n\in\mathbb{N}}$ is a dense sequence of $]0, +\infty[$ and if $(\varepsilon_m)_{m\in\mathbb{N}}$ is a sequence of strictly positive numbers converging to 0, then

$$\mathrm{S}^\nu \subset \bigcap_{\varepsilon>0} \bigcap_{p>0} b^{\eta(p)/p-\varepsilon}_{p,\infty} = \bigcap_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} b^{\frac{\eta(p_n)}{p_n}-\varepsilon_m}_{p_n,\infty}$$

and this inclusion becomes an equality if and only if ν is concave.

Definition

Let $\alpha \in \mathbb{R}$ and $\beta \in \{-\infty\} \cup [0, +\infty[$. A sequence c belongs to the auxiliary space $E(\alpha, \beta)$ if there exists $C, C' \geq 0$ such that

$$\# \left\{ k : |c_{j,k}| \geq C 2^{-\alpha j} \right\} \leq C' 2^{\beta j} \ , \forall j \geq 0.$$

Proposition

For any sequence $(\alpha_n)_{n\in\mathbb{N}}$ dense in $\mathbb{R}$ and $\forall\ (\varepsilon_m)_{m\in\mathbb{N}} \to 0^+$,

$$\mathrm{S}^\nu = \bigcap_{\varepsilon>0} \bigcap_{\alpha\in\mathbb{R}} E(\alpha, \nu(\alpha) + \varepsilon) = \bigcap_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} E(\alpha_n, \nu(\alpha_n) + \varepsilon_m).$$

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The definiton on a distance on every auxiliary spaces $E(\alpha, \nu(\alpha) + \varepsilon)$ will provide a distance on S^ν (initial topology).

Classical result of functional analysis

Let $(E_m)_{m \in \mathbb{N}}$ be a sequence of spaces endowed with the topologies defined by the distances d_m, and let $E = \bigcap_{m \in \mathbb{N}} E_m$. On E, let us consider the topology defined as follows : for every $e \in E$, a basis of neighbourhoods of e is given by the family of sets

$$\bigcap_{(m)} \{f \in E : d_m(e,f) \leq r_m\} \ , \quad r_m > 0.$$

Then,

1. This topology is equivalent to the topology defined on E by the distance d given by

$$d(e,f) = \sum_{m=1}^{+\infty} 2^{-m} \frac{d_m(e,f)}{1 + d_m(e,f)}, \quad e, f \in E.$$

2. A sequence is a Cauchy sequence in (E, d) if and only if it is a Cauchy sequence in (E_m, d_m) for every $m \in \mathbb{N}$.
3. A sequence converges to e in (E, d) if and only if it converges to e in (E_m, d_m) for every $m \in \mathbb{N}$.

Distance on the auxiliary space $E(\alpha, \beta)$

$$\delta_{\alpha,\beta}(c,c') := \inf \left\{ C + C' : C, C' \geq 0 \text{ and } \#E_j(C,\alpha)(c-c') \leq C' 2^{\beta j} \ \forall j \in \mathbb{N} \right\}$$

From the characterization of S^ν as an intersection of auxiliary spaces, we get a distance d on S^ν.

Property

The space (S^ν, d) is a complete separable topological vector space.

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From the characterization of S^{ν} as an intersection of auxiliary spaces, we get a distance d on S^{ν}.

Property

The space (S^{ν}, d) is a complete separable topological vector space.

p local convexity

Definition

Let $0 < p \leq 1$. A subset K of a vector space E is p-convex if for every $x_1, ..., x_N \in K$ and for every $\theta_1, ..., \theta_N \geq 0$ such that $\sum_{n=1}^N \theta_n^p = 1$, then p-convex combinaison $\sum_{n=1}^N \theta_n x_n$ belongs to K. The set K is absolutely p-convex if it is p-convex and if

$$\forall x \in K, \forall |\lambda| \leq 1, \lambda x \in K.$$

Proposition

Let $0 < p \leq 1$. A subset K of a vector space E is absolutely p-convex if and only if

$$\sum_{i=1}^n \mu_i K \subset K$$

for all $\mu_1, ..., \mu_n \in \mathbb{C}$ such that $\sum_{i=1}^n |\mu_i|^p \leq 1$.

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A topological vector space is p-locally convex if it has a basis of 0-nbh absolutely p-convex.

Definition

Let E be a vector space and let $0 < p \leq 1$. An application $q : E \to [0, +\infty[$ is a p semi-norm if

$$\begin{cases} q(\lambda e) = |\lambda| q(e) & \forall \lambda \in \mathbb{C}, e \in E \\ (q(e+f))^p \leq (q(e))^p + (q(f))^p & \forall e, f \in E \end{cases}$$

If moreover, $q(e) = 0 \Leftrightarrow e = 0$, then q is a p norm.

Theorem

A topological vector space $(E, \mathcal{T})$ is p-locally convex if and only if there exists a family of p semi-norms on E that defined a topology equivalent to the topology $\mathcal{T}$.

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We define

$$\underline{\partial}^+\nu(\alpha) := \liminf_{h\to 0^+} \frac{\nu(\alpha+h)-\nu(\alpha)}{h}.$$

for every $\alpha \in \mathbb{R}$ such that $\alpha \geq \alpha_{min}$. The local convexity index of ν is defined by

$$p_0 := \min\left(1, \inf_{\alpha_{min}\leq\alpha<\alpha_{max}} \underline{\partial}^+\nu(\alpha)\right).$$

Proposition

The topological vector space S^ν is not p-normable for any $p > 0$. Moreover,

- if $p_0 < 1$, then S^ν is not p-locally convex for any $p > p_0$;
- if $p_0 > 0$, then S^ν is p_0-locally convex.

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Description of the topology

If $p_0 > 0$, the topology of S^ν is induced by the family of norms $\|.\|_{b^{\alpha_{min}-\varepsilon}_{\infty,\infty}}$ together with the p_0-norms $\|.\|_{\alpha,\varepsilon}$ defined by

$$\|x\|_{\alpha,\varepsilon} := \inf\left\{\|x'\|_{b^s_{p_0,\infty}} + \|x''\|_{b^\alpha_{\infty,\infty}} : x = x' + x''\right\}$$

where $\alpha \in [\alpha_{min}, \alpha_{max}[$, $\varepsilon > 0$ and $s := \alpha + \frac{1-\nu(\alpha)}{p_0} - \varepsilon$. This family of p_0-norms may be made countable by taking a sequence $(\alpha_n)_{n\in\mathbb{N}}$ dense in $[\alpha_{min}, \alpha_{max}[$ and a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ of positive real numbers converging to 0.

Case $p_0 = 0$

If $\alpha_{min} > -\infty$, for every sequence $(p_m)_{m\in\mathbb{N}}$ of $]0,1]$ converging to 0, the topology of the space S^ν can be defined by a sequence of p_m semi-normes. Therefore, the space S^ν is locally pseudoconvex.

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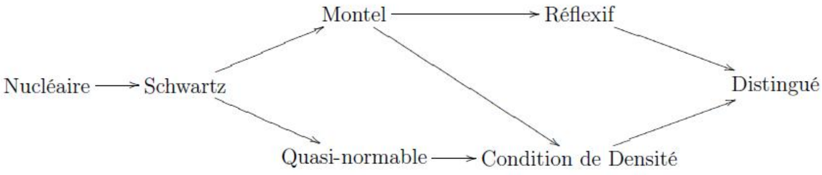
More properties when $p_0 = 1$

- S^ν is a Frechet space
- For Frechet spaces, we have the following relations:



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If $p_0 = 1$, the space S^ν is not nuclear but is a Schwartz space.

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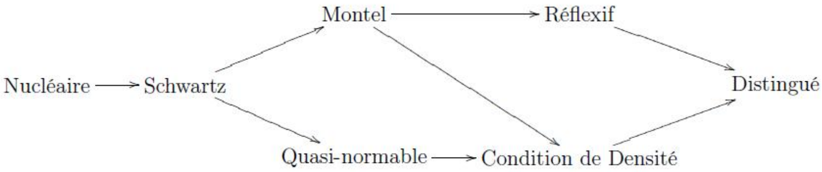

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If $p_0 < 1$?

- The space S^ν is not locally convex
- Generalize the studied properties

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A Hausdorff topological vector space E is **Schwartz** if every 0-nbh U contains a 0-nbh V such that for every $\lambda > 0$, there exists a finite set $M \subset E$ such that

$$V \subset M + \lambda U.$$

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The space S^ν is Schwartz.

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Dual Space

If $u \in (\mathrm{S}^\nu)'$, then u can be identified with a sequence y such that

$$u(x) = \sum_{j \in \mathbb{N}} \sum_{k=0}^{2^j-1} x_{j,k} \overline{y_{j,k}}$$

for every $x \in \mathrm{S}^\nu$. Indeed, we set

$$y = \sum_{j \in \mathbb{N}} \sum_{k=0}^{2^j-1} \overline{u(\vec{e}^{j,k})} e^{j,k}.$$

Notation:

$$[\![\beta]\!] = \begin{cases} -\infty & \text{si } \beta < 0 \\ \beta & \text{si } 0 \leq \beta \leq 1 \\ 1 & \text{si } \beta \geq 1 \end{cases}$$

The dual profile of ν is the function ν' defined on $\mathbb{R}$ by

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for every $x \in \mathrm{S}^\nu$. Indeed, we set

$$y = \sum_{j \in \mathbb{N}} \sum_{k=0}^{2^j-1} \overline{u(\vec{e}^{j,k})} e^{j,k}.$$

Notation:

$$[\![\beta]\!] = \begin{cases} -\infty & \text{si } \beta < 0 \\ \beta & \text{si } 0 \leq \beta \leq 1 \\ 1 & \text{si } \beta \geq 1 \end{cases}$$

The dual profile of ν is the function ν' defined on $\mathbb{R}$ by

$$\nu' : \alpha' \mapsto [\![\alpha' + \inf\{\alpha : \nu(\alpha) - \alpha > \alpha'\}]\!].$$

Proposition

For every decreasing sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ that converges to 0, the dual of S^ν is

$$(\mathrm{S}^\nu)' = \bigcup_{\varepsilon>0} S^{\nu'_\varepsilon} = \bigcup_{m\in\mathbb{N}} S^{\nu'_{\varepsilon_m}}$$

where

$$\nu'_\varepsilon(\alpha') := \nu'(\alpha' - \varepsilon) \ \ \forall \alpha' \in \mathbb{R}\,.$$

Proposition

On the dual space, the strong topology and the inductive limit topolofy coincide.

Wavelet leaders

Standard notation: For $j \in \mathbb{N}_0$, $k \in \{0, \dots, 2^j - 1\}$,

$$\lambda(j,k) := \{x \in \mathbb{R} : 2^j x - k \in [0,1[\} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right[,$$

and for all $j \in \mathbb{N}_0$, Λ_j denote the set of all dyadic interval (of $[0,1[$) of length 2^{-j}.

Definition

The wavelet leaders of a signal $f \in L^2([0,1])$ are defined by

$$d_\lambda := \sup_{\lambda' \subset \lambda} |c_{\lambda'}|, \quad \lambda \in \Lambda_j,\ j \in \mathbb{N}_0$$

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Definition

Let ν be an admissible profile. A sequence c of Ω belongs to $\widetilde{\mathcal{S}}^\nu$ if for every $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$, there exists $J \in \mathbb{N}_0$ such that

$$\#\widetilde{E_j}(C,\alpha)(c) \leq 2^{(\nu(\alpha)+\varepsilon)j} \; \forall j \geq J,$$

where

$$\widetilde{E_j}(C,\alpha)(c) := \left\{k \in \{0,\dots,2^j-1\} : d_{j,k} \geq C2^{-\alpha j}\right\}.$$

For every $c \in \Omega$, we define the asymptotic leader profile of c by

$$\widetilde{\nu}_c(\alpha) := \lim_{\varepsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\ln(\#\widetilde{E_j}(1,\alpha+\varepsilon)(c))}{\ln(2^j)} \right) \right), \quad \alpha \in \mathbb{R}.$$

Proposition

The space $\widetilde{\mathcal{S}}^\nu$ is a linear space and

$$\widetilde{\mathcal{S}}^\nu = \{c \in \Omega : \widetilde{\nu}(\alpha) \leq \nu(\alpha) \; \forall \alpha \in \mathbb{R}\}.$$

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Of course, $\widetilde{\mathcal{S}^\nu} \subset \mathrm{S}^\nu$: sufficient condition on ν to have a strict inclusion, sufficient condition on ν to have an equality + generic results.

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