Fuzzy metrics for switching filters

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Colour image processing has received so much attention in the recent years.

Early approaches process colour images by straightforwardly applying gray-scale methods to each colour channel.

However, many deficiencies arise from this way of processing mainly due to the fact that inter-channel interactions are not taken into account.

Vector Median Filter (Astola et al., 1990)

One of the most well-known filters of this family which are based on vector ordering is the *vector median filter* (VMF) where the colour vectors are ranked using the reduced ordering principle by means of a suitable distance or similarity measure.

The lowest ranked vectors are those which are close to all the other vectors in the window according to the distance or similarity measure used. On the other hand, atypical vectors, susceptible to be considered as noisy or outliers, occupy the highest ranks.

The output of this filter is defined as the lowest ranked vector as follows.

Let $\mathbf{F}$ represent a multichannel image and let W be a window of size $n+1$ (filter length).

The image vectors in the filtering window W are denoted as $\mathbf{F}_j,\ j=0,1,\ldots,n$.

The distance between two vectors $\mathbf{F}_k$ and $\mathbf{F}_j$ is denoted as $\rho(\mathbf{F}_k,\mathbf{F}_j)$, where ρ is a chosen metric.

For each vector in the filtering window, a global or accumulated distance to all the other vectors in the window has to be calculated.

The scalar quantity $R_k = \sum_{j=0, j\neq k}^{n} \rho(\mathbf{F}_k,\mathbf{F}_j)$, is the accumulated distance associated to the vector $\mathbf{F}_k$.

The ordering of the R_k's: $R_{(0)}, R_{(1)}, \ldots, R_{(n)}$, implies the same ordering of the vectors $\mathbf{F}_k$'s: $\mathbf{F}_{(0)}, \mathbf{F}_{(1)}, \ldots, \mathbf{F}(n)$.

Given this order, the output of the filter is $\mathbf{F}_{(0)}$.

Inconveniences of VMF

The inconvenience of the vector median type techniques is that output some of the input vectors for all the pixels in the image (corrupted or non-corrupted).

Switching filters

In order to address this drawback, several approaches have been introduced by using different criteria to preserve the original signal structures, such as edges and fine details.

Most of these filters work in two steps:

- A first step to classify whether a pixel is corrupted or not (detection step).
- A second step to replace only the pixels detected as corrupted.

These methods have been called *switching filters*.

Peer group techniques (Smolka, 2005)

In particular, the notion of *peer group* is used in the switching filters to detect the noise-likely pixels.

This method consists on the construction of the set of the pixels in a filtering window which are close to the central one. The central pixel will be classified as corrupted if its peer group is small and it will be classified as noise-free if its peer group is great enough by means of a parameter that determines whether the peer group is small or not.

Rank-Order Difference technique (Garnett, 2005)

Another notion that has been used in the recent literature is the Rank-Ordered Differences (*mROD*) statistic.

In this case, the detection process consists on the calculation of the sum of the lower *m* distances between the central pixel in the filtering window and the rest of pixels.

$$mROD = \sum_{i=0}^{m} \rho(\mathbf{F}_{(0)}, \mathbf{F}_{(i)})$$

The *mROD* statistic provides a measure of how close a pixel is to its *m* most similar neighbours in *W* attending to their *RGB* colour vectors.

A new statistic based on *ROD*

As a consequence of the ROD statistic notion, a new idea has been implemented in a recent work.

In front of the calculation of the *m* closest distances, one can consider only the *m*-th closest distance to decide whether a pixel is noise-free or not. In this case, it is not important to consider how close the pixels are to the central one except the *m*-th.

$$ROD_m = \rho(\mathbf{F}_{(0)}, \mathbf{F}_{(m)})$$

Metrics

To measure similarity (or difference) between pixels one can use the so called metrics and the most used ones are L_1, L_2 and L_∞.

- The metric L_1 takes into account the noise appearance in all the vector components.
- The metric L_2 works as the L_1 but its efficiency is computationally higher.
- The metric L_∞ takes into account the noise appearance in only one component.

Nevertheless, recently fuzzy metrics have been introduced to extend the classical notions to the fuzzy setting.

Definition (George and Veeramani, 1994)

A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times \mathbb{R}^+$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$

(GV2) $M(x, y, t) = 1$ if and only if $x = y$

(GV3) $M(x, y, t) = M(y, x, t)$

(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(GV5) $M(x, y, \_) : \mathbb{R}^+ \to]0, 1]$ is continuous

The value $M(x, y, t)$ represents the degree of nearness between x and y (with respect to the parameter t) and according to (GV2) $M(x, y, t)$ is close to 0 when x is far from y.

The three most commonly used continuous t-norms in fuzzy logic are:

- The minimum, denoted by $\wedge$
- The usual product, denoted by $\cdot$
- The Lukasievicz t-norm, denoted by $\mathfrak{L}$, defined by

$$x \mathfrak{L} y = \max\{0, x + y - 1\}$$

Proposition

They satisfy the following inequalities:

$$x \mathfrak{L} y \leq x \cdot y \leq x \wedge y$$

$$x * y \leq x \wedge y$$

for each (continuous) t-norm $*$.

Definition (Gregori and Romaguera, 2004)

A fuzzy metric M on X is said to be *stationary* if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant.

In this case we write $M(x, y)$ instead of $M(x, y, t)$.

The interest of fuzzy metrics is mainly due to the following two main advantages with respect to classical metrics:

First, values given by fuzzy metrics are in the interval]0,1] regardless the nature of the distance concept being measured. This implies that it is easy to combine different distance criteria that may originally be in quite different ranges but fuzzy metrics take to a common range. In this way, the combination of several distance criteria may be done in a straightforward way.

Second, fuzzy metrics match perfectly with the employment of other fuzzy techniques since the value given by a fuzzy metric can be directly employed or interpreted as a fuzzy certainty degree of nearness. This allows to straightforwardly include fuzzy metrics as part of other complex fuzzy systems.

Normalizing metrics in the range [0, 1]

Notice that a metric space (X, d) can be normalized to take values in the range [0,1]. It would be enough to take the values

$$d^* = \frac{d}{K + d}$$

Nevertheless, to measure similarity, accordingly with the fuzzy theory, one should consider value

$$1 - d^* = \frac{K}{K + d^*}$$

that coincides with a stationary version of the well-known standard fuzzy metric.

Normalizing metrics in the range [0, 1]

In the case of a bounded metric space (X, d), it can be normalized to take values in the range [0,1] by the expression

$$d^* = \frac{d}{K}$$

where K is an upper bound and accordingly to the fuzzy theory one should consider the value $1 - d^*$ which coincides with an example of fuzzy metric deduced from a metric.

Recently, fuzzy metrics have been implemented in several types of filtering processes with a better performance in the quality. In fact, it has been introduced in the literature a fuzzy metric to simultaneously measure colorimetric difference and spatial distance.

Colorimetric similarity

To determine colorimetric similarity between two pixels it has been used the so called quotient fuzzy metric in three dimensions (with $* = \cdot$), defined by

$$M_q(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{3} \frac{\min\{x_i, y_i\} + K}{\max\{x_i, y_i\} + K}$$

where $K > 0$ is a parameter which can be adjusted attending to each image characteristics (in general, it is used the value $K = 1024$ to guarantee certain symmetry in the *RGB* domain).

Spatial closeness

To evaluate spatial closeness between two pixel in the image matrix, the most used fuzzy metric is the so called standard fuzzy metric [**?**] for a parameter t fixed at $K > 0$, defined by

$$M_d(\mathbf{x}, \mathbf{y}) = \frac{K}{K + d(\mathbf{x}, \mathbf{y})}$$

where d is a classical metric (usually some of the above commented).

A mixed fuzzy metric

So, attending to the fuzzy metrics properties, one can define a new fuzzy metric by the expression

$$M(\mathbf{x},\mathbf{y}) = M_q(\mathbf{x},\mathbf{y}) \cdot M_d(\mathbf{x},\mathbf{y})$$

to evaluate, simultaneously, colorimetric difference and spatial distance.

The M_q fuzzy metric has the important handicap that it is so *drastic* when measuring similarity between two pixels.

First, if $K = 1024$, the M_q distance between the pixels $\mathbf{x} = (0, 0, 0)$ and $\mathbf{y} = (255, 255, 255)$ is $M_q(\mathbf{x}, \mathbf{y}) = \left(\frac{1024}{1279}\right)^3 \approx 0.5132$ and so the range $[0, 1]$ is significantly reduced.

Second, an extreme value in one only component will decrease significantly the corresponding final product and, in consequence, both pixels will be considered very different.

For example, in a gray-scale system, the value 90 could be considered close to 100 and the fuzzy distance with M_q in one dimension is 0.9 (without considering the correction K parameter).

Nevertheless, in the three channel colour space *RGB*:
The fuzzy distance M_q between the pixels $\mathbf{x} = (90, 90, 90)$ and $\mathbf{y} = (100, 100, 100)$ decreases to $0.9^3 = 0.729$.

The fuzzy distance M_q between the pixels $\mathbf{x} = (50, 90, 90)$ and $\mathbf{y} = (100, 100, 100)$ decreases to $0.5 \cdot 0.9^2 = 0.405$.

We propose to consider the following similarity function:

$$M_1(\mathbf{x}, \mathbf{y}) = \frac{\sum_{i=1}^{3} \min\{x_i, y_i\}}{\sum_{i=1}^{3} \max\{x_i, y_i\}}$$

Notice that, in this case, the colorimetric difference between the previously considered pixels is

$$M_1((90, 90, 90), (100, 100, 100)) = 0.9$$

$$M_1((50, 90, 90), (100, 100, 100)) = 0.767$$

that can be considered to better agree with the gray-scale case.

This function will compare similarity between pixels with a better sensibility and not to be so drastic.

Proposition

The similarity measure M_1 satisfies the following conditions:

(i) $M_1(\mathbf{x},\mathbf{y}) \geq 0$

(ii) $M_1(\mathbf{x},\mathbf{y}) = M(\mathbf{y},\mathbf{x})$

(iii) $M_1(\mathbf{x},\mathbf{y}) = 1$ if and only if $\mathbf{x} = \mathbf{y}$

It is an open problem if M_1 is a fuzzy metric on $\mathbb{N}^n$ in the sense of George and Veeramani, for the Lukasievicz t-norm, i.e. if M_1 satisfies the triangle inequality $M_1(\mathbf{x}, \mathbf{z}) \geq M_1(\mathbf{x}, \mathbf{y}) * M_1(\mathbf{y}, \mathbf{z})$ for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{N}^n$.

We have seen that M_1 is not a fuzzy metric for the usual product in the case of $\mathbb{N}^2$. Indeed, consider the following colour pixels (in a two-space range): $\mathbf{x} = (1,6)$, $\mathbf{y} = (5,8)$ and $\mathbf{z} = (7,2)$. Then we have

$$M_1(\mathbf{x},\mathbf{z}) = \frac{1+2}{7+6} = \frac{3}{13}$$

$$M_1(\mathbf{x},\mathbf{y}) \cdot M_1(\mathbf{y},\mathbf{z}) = \frac{1+6}{5+8} \cdot \frac{5+2}{7+8} = \frac{7}{13} \cdot \frac{7}{15} = \frac{49}{195} > \frac{3}{13}$$