Kannan mappings vs. Caristi mappings: An easy example

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In 1922, Banach published his famous fixed point theorem wich is stated as follows.

Theorem 1 (Banach). *Let (X, d) be a complete metric space. If T is a self-mapping of X such that there is a constant $c \in [0, 1)$ satisfying*

$$d(Tx, Ty) \leq cd(x, y), \tag{1}$$

for all $x, y \in X$, then T has a unique fixed point.

Kannan (1968) proved the following fixed point theorem which is independent from Banach's fixed point theorem.

Theorem 2 (Kannan). *Let (X, d) be a complete metric space. If T is a self-mapping of X such that there is a constant $c \in [0, 1/2)$ satisfying*

$$d(Tx, Ty) \leq c(d(x, Tx) + d(y, Ty)), \tag{2}$$

for all $x, y \in X$, then T has a unique fixed point.

Later on, Chatterjea (1971) obtained the following variant of Kannan's fixed point theorem.

Theorem 3 (Chatterjea). *Let (X, d) be a complete metric space. If T is a self-mapping of X such that there is a constant $c \in [0, 1/2)$ satisfying*

$$d(Tx, Ty) \leq c(d(x, Ty) + d(y, Tx)), \tag{3}$$

for all $x, y \in X$, then T has a unique fixed point.

The above results suggest the following well-established notion

Definition 1. Let T be a self-map of a metric space (X, d). Then T is called a Banach contraction (resp. a Kannan mapping, a Chatterjea mapping) if T satisfies condition (1) (resp. condition (2), condition (3)) for all $x, y \in X$.

Contrarily to the Banach contractions, not every Kannan mapping is a continuous mapping and not every Chatterjea mapping is a continuous mapping.

On the other hand, Banach's fixed point theorem does not characterize metric completeness. Indeed, there exist examples of non complete metric spaces for which every Banach contraction has a fixed point. However, both Kannan's fixed point theorem and Chatterjea's fixed point theorem characterize metric completeness, as Subrahmanyam (1975) showed.

Theorem 4 (Subrahmanyam). *For a metric space* (X, d) *the following conditions are equivalent.*
(1) (X, d) *is complete.*
(2) *Every Kannan mapping on* X *has a fixed point.*
(3) *Every Chatterjea mapping on* X *has a fixed point.*

In his well-known paper published in 1976, Caristi proved the following important fixed point theorem that also allows to characterize metric completeness and is "equivalent" to the Ekeland Variational Principle.

Theorem 5 (Caristi). *Let* (X, d) *be a complete metric space. If* T *is a self-mapping of* X *such that there is a lower semicontinuous function* $\varphi : X \to [0, \infty)$ *satisfying*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \tag{4}$$

for all $x \in X$*, then* T *has a fixed point.*

A self-mapping T on a metric space (X, d) for which there is a lower semicontinuous function $\varphi : X \to [0, \infty)$ satisfying condition (4) for all $x \in X$ is called a Caristi mapping.
Kirk (1976) proved the "if" part of the following characterization.

Theorem 6 (Kirk). *A metric space (X, d) is complete if and only if every Caristi mapping on X has a fixed point.*

The relationship between Banach mappings and Caristi mappins, as well as between Kannan mappings (resp. Chatterjea mappings) and Caristi mappings has been considered by several authors. Thus, following a construction suggested by Weston (1977), Park (1984) asserted that if T is a Chatterjea mapping on a metric space (X, d), with constant $c \in [0, 1/2)$, then the function $\varphi : X \to [0, \infty)$ defined as

$$\varphi(x) = \frac{1-c}{1-2c} d(x, Tx), \tag{5}$$

for all $x \in X$, is a Caristi mapping, and, hence, every Chatterjea mapping is a Caristi mapping.

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CHARACTERIZATIONS OF METRIC COMPLETENESS

BY

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In this paper, we give some necessary and sufficient conditions for a metric space (X, d) to be complete. Such characterizations of metric completeness are given mainly by results relevant to Caristi's fixed point theorem. Works of Cantor, Kuratowski, Ekeland, Caristi, Kirk, Wong, Weston, Ćirić, Hu, Reich, Subrahmanyam, and others are combined.

Kuratowski [18] first noticed that the Cantor intersection theorem characterizes the metric completeness. Hu [12] showed that a metric space is complete if and only if any Banach contraction on closed subsets thereof has a fixed point. On the other hand, Kirk [15] showed that Caristi's theorem characterizes the metric completeness. Later, motivated by Wong's proof [27] of Caristi's theorem, Weston [26] showed that a metric space X is complete if and only if X satisfies a condition of Ekeland [10], [11], that is, for each lower semicontinuous function $h: X \to (-\infty, \infty)$ bounded from below on X, there is a point p in X such that $h(p)-h(x) < d(p, x)$ for every point x in X. Reich [23] and Subrahmanyam [25] also obtained characterizations of the metric completeness using Kannan's result [13] similar to the Banach contraction principle, which is known to be a consequence of Caristi's theorem. On the other hand, Kolodner [17] and Boyd and Wong [2] noticed that the Banach contraction principle follows from the Cantor intersection theorem.

Now we combine those results and state our characterizations of the metric completeness. Let ω denote the set of nonnegative integers and $^-$ the closure operation.

THEOREM. *For a metric space (X, d), the following statements are equivalent*:

(i) *X is complete.*

(ii) *For every sequence $\{\alpha_n\}_{n\in\omega}$ of positive numbers converging to* 0 *and every sequence $\{F_n\}_{n\in\omega}$ of nonempty closed subsets of X such that $F_{n+1} \subset F_n$, $n\in\omega$, and each F_n is a union of finite number of subsets of diameter less than α_n, we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

(iii) *For every sequence* $\{F_n\}_{n\in\omega}$ *of nonempty closed subsets of* X *such that* $F_{n+1} \subset F_n$, $n\in\omega$, *and the sequence* $\{\operatorname{diam} F_n\}_{n\in\omega}$ *converges to* 0, *we have* $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

(iv) *Every lower semicontinuous function* $h\colon X \to (-\infty, \infty)$ *which is bounded from below has a d-point* p *in* X, *that is,*

$$h(p) - h(x) < d(p, x)$$

for every point x *in* X, $x \neq p$.

(v) *For every selfmap* f *of* X *with a lower semicontinuous function* $V\colon X \to (-\infty, \infty)$ *which is bounded from below and such that, for each* x *in* X *with* $x \neq fx$, *there exists* y *in* $X - \{x\}$ *satisfying*

$$d(x, y) \leqslant V(x) - V(y),$$

f *has a fixed point.*

(vi) *For every selfmap* f *of* X *such that there exists a lower semicontinuous function* $\varphi\colon X \to (-\infty, \infty)$ *which is bounded from below and satisfies*

$$d(x, fx) \leqslant \varphi(x) - \varphi(fx)$$

for each x *in* X, f *has a fixed point.*

(vii) *For every selfmap* f *of* X *such that there exist a* $u\in X$ *and an* $\alpha\in[0, 1)$ *satisfying*

$$d(fx, f^2 x) \leqslant \alpha d(x, fx)$$

for each x *in* $\overline{\{f^n u\}}_{n\in\omega}$ *and* f *is continuous on* $\overline{\{f^n u\}}$, f *has a fixed point in* $\overline{\{f^n u\}}$.

(viii) *For every selfmap* f *of* X *such that there exist a* $u\in X$ *and an* $\alpha\in[0, 1)$ *satisfying*

$$d(fx, fy) \leqslant \alpha \max\{d(x, y),\ d(x, fx),\ d(y, fy),\ [d(x, fy) + d(y, fx)]/2\}$$

for all $x, y\in\overline{\{f^n u\}}$, f *has a (unique) fixed point in* $\overline{\{f^n u\}}$.

Proof. (i) $\Rightarrow$ (ii) is given in [18], [19], and (ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (iv). We order X by defining $x \leqslant y$ iff $d(x, y) \leqslant h(x) - h(y)$. For each $x\in X$, let $X(x) = \{y\in X \mid x \leqslant y\}$. We construct an increasing sequence $\{x_n\}$ as follows: Choose $x_0\in X$ arbitrarily, and if $x_0, \ldots, x_n$ are given, then choose $x_{n+1}\in X(x_n)$ with $h(x_{n+1}) < \inf h(X(x_n)) + 1/n$. Thus $x_n \leqslant x_{n+1}$ and for each $x\in X(x_{n+1}) \subset X(x_n)$ we have

$$h(x_{n+1}) - 1/n < \inf h\big(X(x_n)\big) \leqslant h(x)$$

and

$$d(x, x_{n+1}) \leqslant h(x_{n+1}) - h(x).$$

is a Banach contraction on $\overline{\{f^n x_0\}}$; hence f satisfies the hypothesis of (viii) on $\overline{\{f^n x_0\}}$. However, f does not have a fixed point.

This completes our proof.

Remarks. (1) Kuratowski [18] obtained (i) $\Rightarrow$ (ii) as a generalization of the Cantor intersection theorem (i) $\Rightarrow$ (iii) (see [19]). He also noticed that (i) $\Leftrightarrow$ (iii).

(2) (i) $\Rightarrow$ (iv) was actually due to Ekeland [10], [11]. Weston proved (i) $\Leftrightarrow$ (iv). Our proof of (iii) $\Rightarrow$ (iv) is based on the proof of (i) $\Rightarrow$ (vi) of Penot [22].

(3) Caristi's fixed point theorem (i) $\Rightarrow$ (vi) with $\varphi\colon X \to [0, \infty)$ was given in [7]. It is actually equivalent to (i) $\Rightarrow$ (iv) announced in 1972 by Ekeland [10], whose result is an abstraction of a lemma due to Bishop and Phelps [1]. Various proofs of Caristi's theorem were given by Brønsted [4], [5], Browder [6], Kasahara [14], Kirk [15], Pasicki [21], Penot [22], Siegel [24], and Wong [27]. Condition (vi) was due to Brønsted [5].

(4) Kirk [15] showed (i) $\Leftrightarrow$ (vi). Wong [27] claimed that (i) $\Rightarrow$ (vi) implies (i) $\Rightarrow$ (v). A proof of (v) $\Rightarrow$ (iv) was also given by Wong [28]. Brézis and Browder [3] showed that (i) $\Rightarrow$ (vi) is equivalent to (i) $\Rightarrow$ (iv).

(5) Kirk and Caristi [16] noted that (i) $\Rightarrow$ (vi) implies the Banach contraction principle. Weston [26] noted that, by putting

$$\varphi(x) = (1-2\alpha)^{-1}(1-\alpha)\,d(x, fx),$$

the contractive type condition

$$d(fx, fy) \leqslant \alpha\,\{d(x, fy) + d(y, fx)\}, \qquad \alpha < 1/2,$$

implies the hypothesis of (vi).

(6) In the proof of (vi) $\Rightarrow$ (vii), f and φ are continuous on $\overline{\{f^n u\}}$. Browder [6] observed that, in (i) $\Rightarrow$ (vi), if f is continuous, then $\lim f^n x$ exists for all $x \in X$ and it is fixed under f.

(7) A variant of (i) $\Rightarrow$ (viii) was first given by Ćirić [8], and later extended by a number of authors. Pal and Maiti [20] considered an extended form of (viii), which is a particular case of (vii).

(8) The basic idea of the proof of (viii) $\Rightarrow$ (i) is due to Hu [12]. Reich [23] used Hu's idea with respect to Kannan's contractive condition [13]:

$$d(fx, fy) \leqslant \alpha\,\{d(x, fx) + d(y, fy)\}, \qquad \alpha < 1/2.$$

Similar results are given also by Subrahmanyam [25].

(9) Kolodner [17] and Boyd and Wong [2] noticed that the Cantor intersection theorem implies the Banach contraction principle. However, using an example of Connell [9], Subrahmanyam [25] noticed that the

In this direction, Shioji, Suzuki and Takahashi (1998), and Petrusel (2003) claim that every Banach contraction and every Kannan mapping on a metric space is a Caristi mapping. In fact (see e.g. Petrusel (2003)) if T is a self-mapping on a metric space (X, d) the following facts are asserted:

(A) If T is a Banach contraction, with constant $c \in [0, 1)$, then the function $\varphi : X \to [0, \infty)$ defined as

$$\varphi(x) = \frac{1}{1 - c} d(x, Tx), \tag{6}$$

for all $x \in X$, is a Caristi mapping on X.

(B) If T is a Kannan mapping, with constant $c \in [0, 1/2)$, then the function $\varphi : X \to [0, \infty)$ defined as in (5) for all $x \in X$, is a Caristi mapping on X.

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CONTRACTIVE MAPPINGS, KANNAN MAPPINGS AND METRIC COMPLETENESS

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ABSTRACT. In this paper, we first study the relationship between weakly contractive mappings and weakly Kannan mappings. Further, we discuss characterizations of metric completeness which are connected with the existence of fixed points for mappings. Especially, we show that a metric space is complete if it has the fixed point property for Kannan mappings.

1. INTRODUCTION

Let X be a metric space with metric d. Then a function p from $X \times X$ into $[0, \infty)$ is called a w-distance on X if it satisfies the following:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
(2) p is lower semicontinuous in its second variable;
(3) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The concept of a w-distance was first introduced by Kada, Suzuki and Takahashi [6]. They give some examples of w-distance and improved Caristi's fixed point theorem [2], Ekeland's variational principle [4] and the nonconvex minimization theorem according to Takahashi [12]. We denote by $W(X)$ the set of all w-distances on X. A mapping T from X into itself is called weakly contractive [11] if there exist $p \in W(X)$ and $r \in [0, 1)$ such that

$$p(Tx, Ty) \leq rp(x, y) \quad \text{for all} \quad x, y \in X.$$

In particular, if $p = d$, T is called contractive. Suzuki and Takahashi [11] proved that a metric space is complete if and only if it has the fixed point property for weakly contractive mappings. A mapping T from X into itself is called weakly Kannan [10] if there exist $p \in W(X)$ and $\alpha \in [0, 1/2)$ such that

$$p(Tx, Ty) \leq \alpha\big(p(Tx, x) + p(Ty, y)\big) \quad \text{for all} \quad x, y \in X,$$

or

$$p(Tx, Ty) \leq \alpha\big(p(Tx, x) + p(y, Ty)\big) \quad \text{for all} \quad x, y \in X.$$

In particular, if $p = d$, T is called Kannan [7]. Suzuki [10] proved that a complete metric space has the fixed point property for weakly Kannan mappings. On the

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other hand, characterizations of metric completeness have been discussed by many authors (cf. [3, 5, 8, 9, 12]). It has been known that the fixed point property for contractive mappings does not characterize metric completeness. For example, see [11]. But Hu [5] proved that a metric space is complete if every closed subspace has the fixed point property for contractive mappings. Reich [9] also proved that a metric space is complete if every closed subspace has the fixed point property for Kannan mappings. We recall that a mapping T from a metric space X into itself is said to be Caristi if there exists a lower semicontinuous function φ from X into $[0,\infty)$ such that $d(x,Tx) \le \varphi(x) - \varphi(Tx)$ for all $x \in X$. Note that Caristi mappings include Kannan mappings and contractive mappings. Kirk [8] proved that a metric space is complete if it has the fixed point property for Caristi mappings. Thus Caristi mappings characterize metric completeness whereas contractive mappings do not. This leaves open the question whether Kannan mappings characterize metric completeness or not.

In this paper, we first study the relationship between weakly contractive mappings and weakly Kannan mappings. Further, we discuss characterizations of metric completeness which are connected with the existence of fixed points for mappings. Especially, we show that a metric space is complete if it has the fixed point property for Kannan mappings.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the sets of positive integers, integers, rational numbers and real numbers, respectively.

Let X be a metric space with metric d. A w-distance p on X is called symmetric if $p(x,y) = p(y,x)$ for all $x, y \in X$. We denote by $W_0(X)$ the set of all symmetric w-distances on X. Note that the metric d is an element in $W_0(X)$. We denote by $WC_1(X)$ the set of all mappings T from X into itself such that there exist $p \in W(X)$ and $r \in [0,1)$ satisfying

$$p(Tx,Ty) \le rp(x,y) \quad \text{for all} \quad x,y \in X,$$

i.e., the set of all weakly contractive mappings from X into itself. We define the sets $WC_2(X)$, $WC_0(X)$, $WK_1(X)$, $WK_2(X)$ and $WK_0(X)$ of mappings from X into itself as follows: $T \in WC_2(X)$ if and only if there exist $p \in W(X)$ and $r \in [0,1)$ such that

$$p(Tx,Ty) \le rp(y,x) \quad \text{for all} \quad x,y \in X;$$

$T \in WC_0(X)$ if and only if there exist $p \in W_0(X)$ and $r \in [0,1)$ such that

$$p(Tx,Ty) \le rp(x,y) \quad \text{for all} \quad x,y \in X;$$

$T \in WK_1(X)$ if and only if there exist $p \in W(X)$ and $\alpha \in [0,1/2)$ such that

$$p(Tx,Ty) \le \alpha\big(p(Tx,x) + p(Ty,y)\big) \quad \text{for all} \quad x,y \in X;$$

$T \in WK_2(X)$ if and only if there exist $p \in W(X)$ and $\alpha \in [0,1/2)$ such that

$$p(Tx,Ty) \le \alpha\big(p(Tx,x) + p(y,Ty)\big) \quad \text{for all} \quad x,y \in X;$$

$T \in WK_0(X)$ if and only if there exist $p \in W_0(X)$ and $\alpha \in [0,1/2)$ such that

$$p(Tx,Ty) \le \alpha\big(p(Tx,x) + p(Ty,y)\big) \quad \text{for all} \quad x,y \in X.$$

We recall T is weakly Kannan if $T \in WK_1(X) \cup WK_2(X)$.

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CARISTI TYPE OPERATORS AND APPLICATIONS

ADRIAN PETRUŞEL

Dedicated to Professor Gheorghe Micula at his 60^{th} *anniversary*

1. Introduction

Caristi's fixed point theorem states that each operator f from a complete metric space (X, d) into itself satisfying the condition:

there exists a proper lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that:

$$d(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \text{ for each } x \in X \tag{1.1}$$

has at least a fixed point $x^* \in X$, i. e. $x^* = f(x^*)$. (see Caristi [4]).

For the multi-valued case, there exist several results involving multi-valued Caristi type conditions. For example, if F is a multi-valued operator from a complete metric space (X, d) into itself and if there exists a proper, lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\text{for each } x \in X, \text{ there is } y \in F(x) \text{ so that } d(x, y) + \varphi(y) \leq \varphi(x), \tag{1.2}$$

then the multi-valued map F has at least a fixed point $x^* \in X$, i. e. $x^* \in F(x^*)$. (see Mizoguchi-Takahashi [11]).

Moreover, if F satisfies the stronger condition:

$$\text{for each } x \in X \text{ and each } y \in F(x) \text{ we have } d(x, y) + \varphi(y) \leq \varphi(x), \tag{1.3}$$

then the multi-valued map F has at least a strict fixed point $x^* \in X$, i. e. $\{x^*\} = F(x^*)$. (see Maschler-Peleg [10]).

Another result of this type was proved by L. van Hot, as follows.

If F is a multi-valued operator with nonempty closed values and $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semi-continuous function such that the following condition holds:

$$\text{for each } x \in X, \ \inf \{ \ d(x, y) + \varphi(y) \ : \ y \in F(x) \ \} \leq \varphi(x), \tag{1.4}$$

then F has at least a fixed point. (see van Hot [6])

There are several extensions and generalizations of these important principles of nonlinear analysis (see the references list and also the bibliography therein)

The purpose of this paper is to present several new results and open problems for single-valued and multi-valued Caristi type operators between metric spaces. Also,

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Then a function $p : X \times X \to \mathbb{R}_+$, *defined by*

$$p(x,y) := |\int_x^y f(u)du|, \text{ for every } x, y \in X$$

is a w-distance on X.

For other examples and related results, see Kada, Suzuki and Takahashi [7].

Some important properties of the w-distance are contained in:

Lemma 2.1. *Let* (X, d) *be a metric space and* p *be a w-distance on* X. *Let* $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ *be sequences in* X, *let* $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ *be sequences in* $\mathbb{R}_+$ *converging to* 0 *and let* $y, z \in X$. *Then the following hold:*

(i) if $p(x_n, x_m) \leq \alpha_n$, *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* (x_n) *is a Cauchy sequence.*

(ii) if $p(y, x_n) \leq \alpha_n$, *for any* $n \in \mathbb{N}$, *then* (x_n) *is a Cauchy sequence.*

(iii) if $p(x_n, y_n) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$, *for any* $n \in \mathbb{N}$ *then* (y_n) *converges to* z.

3. Single-valued Caristi type operators

If $f : X \to X$ is an a-contraction, then it is well-known (see for example Dugundji-Granas [5]) that f is a Caristi type operator with a function $\varphi(x) = \frac{1}{1-a}d(x, f(x))$. Also, Caristi type mappings include Reich type operators and in particular Kannan operators. Indeed, if f satisfies a Reich type condition with constants a, b, c, then f is a Caristi type operator with a function $\varphi(x) = \frac{1-c}{1-a-b-c}d(x, f(x))$.

Moreover, if $f : X \to X$ satisfies the following condition (see I. A. Rus (1972), [14]):

there is $a \in [0, 1[$ such that $d(f(x), f^2(x)) \leq ad(x, f(x))$, for each $x \in X$

then f is a Caristi operator with a function $\varphi(x) = \frac{1}{1-a}d(x, f(x))$.

Hence, the class of single-valued Caristi type operators is very large, including at least the above mentioned types of contractive mappings.

Some characterizations of metric completeness have been discussed by several authors such as Weston, Kirk, Suzuki, Suzuki and Takahashi, Shioji, Suzuki and Takahashi, etc. For example, Kirk [8] proved that a metric space is complete if it has the fixed point property for Caristi mappings. Moreover, Shioji, Suzuki and Takahashi proved in [15] that a metric space is complete if and only if it has the fixed point property for Kannan mappings. On the other hand, it is well-known that the fixed point property for a-contraction mappings does not characterize metric completeness, see for example Suzuki-Takahashi [16]. Thus, Kannan mappings and Caristi mappings characterize metric completeness, while contraction mappings do not. Regarding to the problem of characterizations of metric completeness by means of contraction mappings, Suzuki and Takahashi and independently M. C. Anisiu and V. Anisiu showed (see [16] respectively [1]) that a convex subset Y of a normed space is complete if and only if every contraction $f : Y \to Y$ has a fixed point in Y.

The following generalization of Caristi's theorem is proved in Kada-Suzuki-Takahashi [7]:

It is easy to check that assertion (A) is correct. However, in the case that T is a Kannan mapping, or a Chatterjea mapping, on a metric space (X, d), the function φ given by (5) indeed satisfies $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$, but, unfortunately, the function $x \to d(x, Tx)$ is not lower semicontinuous in general, as Example 1 below shows. Therefore, it seems that the following question still is open: Is every Kannan mapping on a metric space X, a Caristi mapping on X ?

Example 1. Let $X = [0, \infty)$ and let d be the usual metric on X. Fix $\delta \in (0, 1)$ and define $T : X \to X$ as

$Tx = 0$ if $x \in [0, 1 - \delta)$;

$Tx = x/4$ if $x \in [1 - \delta, 1)$, and

$Tx = (1 - \delta)/4$ if $x \geq 1$.

Then T is both a Kannan mapping and a Chatterjea mapping on (X, d).

Finally, define $f : X \to [0, \infty)$ as $f(x) = d(x, Tx)$. We show that f is not lower semicontinuous at $x = 1$.
Indeed, choose a sequence $(x_n)_n$ in X such that $x_n \in (1 - \delta, 1)$ and $d(1, x_n) \to 0$. We have

$$\begin{aligned} f(1) - f(x_n) &= d\left(1, \frac{1-\delta}{4}\right) - d\left(x_n, \frac{x_n}{4}\right) = \frac{3+\delta}{4} - \frac{3x_n}{4} \\ &> \frac{3+\delta}{4} - \frac{3}{4} = \frac{\delta}{4}, \end{aligned}$$

for all n, so f is not lower semicontinuous at $x = 1$.