Probabilistic uniform structures

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Definition

A uniformity $\mathcal{U}$ in a nonempty set X is a filter of binary relations on X which satisfy:

- $\Delta \subseteq U$ for all $U \in \mathcal{U}$ where $\Delta = \{(x,x) : x \in X\}$;
- if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$ where $U^{-1} = \{(x,y) \in X \times X : (y,x) \in U\}$;
- if $U \in \mathcal{U}$ we can find $V \in \mathcal{U}$ such that

$$V^2 \subseteq U$$

where $V^2 = \{(x,y) \in X^2 : \exists z \in X \text{ with } (x,z),(z,y) \in V\}$.

Definition

Let X be a nonempty set. A gauge or a uniform structure on X is a nonempty family $\mathcal{D}$ of pseudometrics on X such that:

(G1) if $d, q \in \mathcal{D}$ then $d \vee q \in \mathcal{D}$;

(G2) if e is a pseudometric on X and for each $\varepsilon > 0$ there exist $d \in \mathcal{D}$ and $\delta > 0$ such that $d(x, y) < \delta$ implies $e(x, y) < \varepsilon$ for all $x, y \in X$, then $e \in \mathcal{D}$.

Definition

A function $f : (X, \mathcal{D}) \to (Y, \mathcal{Q})$ between two spaces endowed with a uniform structure is uniformly continuous if
$f : (X, \bigvee_{d\in\mathcal{D}} \mathcal{U}_d) \to (Y, \bigvee_{q\in\mathcal{Q}} \mathcal{U}_q)$ is uniformly continuous

$$\mathsf{Unif} \underset{\Lambda}{\overset{\Delta}{\rightleftarrows}} \mathsf{SUnif}$$

- $\Delta(\mathcal{U}) = \{d \text{ pseudometric } : \mathcal{U}_d \subseteq \mathcal{U}\}$;
- $\Lambda(\mathcal{D}) = \bigvee_{d\in\mathcal{D}} \mathcal{U}_d$.

Definition

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an Abelian topological monoid with unit 1, such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Example

- $a \wedge b = \min\{a, b\}$
- $a \cdot b = ab$
- $a *_L b = \max\{a + b - 1, 0\}$

$$* \leq \wedge \text{ for each continuous t-norm } *.$$

Definition

A fuzzy pseudometric (in the sense of Kramosil and Michalek) on a nonempty set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, +\infty)$ such that for all $x, y, z \in X,\ t, s > 0$:

(FM1) $M(x, y, 0) = 0$;

(FM2) $M(x, x, t) = 1$;

(FM3) $M(x, y, t) = M(y, x, t)$;

(FM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(FM5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous;

If the fuzzy pseudometric $(M, *)$ also satisfies:

(FM2') $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$

then $(M, *)$ is said to be a fuzzy metric on X.

In this case, $(X, M, *)$ is said to be a fuzzy (pseudo)metric space.

Every fuzzy (pseudo)metric $(M, *)$ on X generates a topology $\tau(M)$ on X which has as a base the family $\{B_M(x, \varepsilon, t) : x \in X, 0 < \varepsilon < 1, t > 0\}$, where

$$B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$$

for all $\varepsilon \in (0, 1)$ and $t > 0$.

Proposition (Gregori and Romaguera, 2000)

A fuzzy metric space $(X, M, *)$ is metrizable and the countable family $\{U_n : n \in \mathbb{N}\}$ where

$$U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - 1/n\}$$

forms a base for a compatible uniformity $\mathcal{U}_M$.

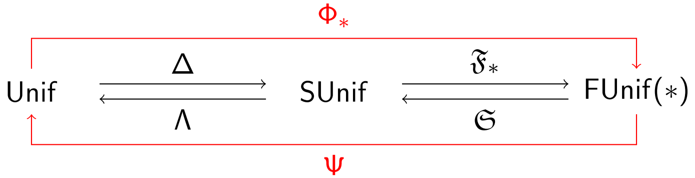
Standard fuzzy pseudometric

Example

Let (X, d) be a pseudometric space. Let M_d be the fuzzy set on $X \times X \times [0, \infty)$ given by

$$M_d(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

For every continuous t-norm $*$, $(M_d, *)$ is a fuzzy pseudometric on X which is called the *standard fuzzy pseudometric* induced by d. Furthermore, we notice that $\mathcal{U}_d = \mathcal{U}_{M_d}$ where $\mathcal{U}_d$ is the uniformity generated by d.



- $\mathfrak{F}_*(\mathcal{D}) = \langle \{M_d : d \in \mathcal{D}\} \rangle$;
- $\mathfrak{S}(\mathcal{M}) = \{d \text{ pseudometric } : \mathcal{U}_d \subseteq \bigvee_{M \in \mathcal{M}} \mathcal{U}_M\}$.

Definition (Gutiérrez García, Romaguera and Sanchis, 2010)

Let X be a nonempty set and let $*$ be a continuous t-norm. A fuzzy uniform structure for $*$ is a pair $(\mathcal{M}, *)$ where $\mathcal{M}$ is a nonempty family of fuzzy pseudometrics with respect $*$ such that:

(FU1) if $(M, *), (N, *) \in \mathcal{M}$ then $(M \wedge N, *) \in \mathcal{M}$;

(FU2) if $(M, *)$ is a fuzzy pseudometric on X, and if for each $\varepsilon \in (0, 1)$ and each $t > 0$ there exist $(N, *) \in \mathcal{M}$, $\delta \in (0, 1)$ and $s > 0$ such that

$$N(x, y, s) \geq 1 - \delta \text{ implies } M(x, y, t) \geq 1 - \varepsilon$$

for all $x, y \in X$, then $(M, *) \in \mathcal{M}$.

A fuzzy uniform space is a triple $(X, \mathcal{M}, *)$ such that X is a nonempty set and $(\mathcal{M}, *)$ is a fuzzy uniform structure on X.

Definition (Gutiérrez García, Romaguera and Sanchis, 2010)

Let $(X, \mathcal{M}, *)$ and $(Y, \mathcal{N}, \star)$ be two fuzzy uniform spaces. A mapping $f : X \to Y$ is said to be uniformly continuous if for each $N \in \mathcal{N}$, $\varepsilon \in (0,1)$ and $t > 0$ there exist $M \in \mathcal{M}$, $\delta \in (0,1)$ and $s > 0$ such that $N(f(x), f(y), t) > 1 - \varepsilon$ whenever $M(x, y, s) > 1 - \delta$.

FUnif $\equiv$ category of fuzzy uniform spaces

Definition (Gutiérrez García, Romaguera and Sanchis, 2010)

Let $(X, \mathcal{M}, *)$ and $(Y, \mathcal{N}, \star)$ be two fuzzy uniform spaces. A mapping $f : X \to Y$ is said to be uniformly continuous if for each $N \in \mathcal{N}$, $\varepsilon \in (0, 1)$ and $t > 0$ there exist $M \in \mathcal{M}$, $\delta \in (0, 1)$ and $s > 0$ such that $N(f(x), f(y), t) > 1 - \varepsilon$ whenever $M(x, y, s) > 1 - \delta$.

FUnif $\equiv$ category of fuzzy uniform spaces

Definition (Höhle 78, Katsaras 79)

A probabilistic uniformity on a nonempty set X is a pair $(\mathcal{U}, *)$, where $*$ is a continuous t-norm and $\mathcal{U}$ is a prefilter on $X \times X$ such that:

(FU1) $U(x,x) = 1$ for all $U \in \mathcal{U}$;

(FU2) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$ where $U^{-1}(x,y) = U(y,x)$;

(FU3) for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$V^2 \leq U$$

where $V^2(x,y) = \sup_{z \in X} V(x,z) * V(z,y)$;

In this case we say that $(X, \mathcal{U}, *)$ is a probabilistic uniform space.

Definition

A function $f:(X,\mathcal{U},*)\to(Y,\mathcal{V},\star)$ between two probabilistic uniform spaces is said to be fuzzy uniformly continuous if for every $V\in\mathcal{V}$ we can find $U\in\mathcal{U}$ such that

$$U(x,y)\leq V(f(x),f(y)) \text{ for all } x,y\in X.$$

PUnif $\equiv$ category of probabilistic uniform spaces

Lowen functors

$\omega_* : \mathsf{Unif} \to \mathsf{PUnif}(*)$

$$\omega_*(\mathcal{U}) = \{F \in I^{X\times X} : F^{-1}(]\varepsilon, 1]) \in \mathcal{U} \text{ for all } \varepsilon \in [0,1[\}$$

$\iota : \mathsf{PUnif} \to \mathsf{Unif}$

$$\iota(\mathfrak{U}) = \{U^{-1}(]\varepsilon, 1]) : U \in \mathfrak{U}, \varepsilon \in [0,1[\}$$

- ω_* and ι are adjoint functors;
- $\iota(\omega_*(\mathcal{U})) = \mathcal{U}$;
- $\mathfrak{U} \subseteq \omega_*(\iota(\mathfrak{U}))$.

Probabilistic uniform structures

Definition

Let X be a nonempty set and let $*$ be a continuous t-norm. A probabilistic $*$-uniform structure on X is a pair $(\mathcal{M}, *)$ where $\mathcal{M}$ is a family of fuzzy pseudometrics on X with respect to $*$ such that:

- if $(M, *), (N, *) \in \mathcal{M}$ then $(M \wedge N, *) \in \mathcal{M}$;
- if $(M, *)$ is a fuzzy pseudometric on X such that for all $t > 0$, there exist $(N, *) \in \mathcal{M}$ and $s > 0$ verifying

$$N(x, y, s) \leq M(x, y, t)$$

  for all $x, y \in X$, then $(M, *) \in \mathcal{M}$.

A space with a probabilistic $*$-uniform structure is a triple $(X, \mathcal{M}, *)$ such that X is a nonempty set and $(\mathcal{M}, *)$ is a probabilistic $*$-uniform structure on X.

Definition

Let $(X, \mathcal{M}, *)$ and $(Y, \mathcal{N}, \star)$ be two spaces endowed with two probabilistic uniform structures. A mapping $f : X \to Y$ is said to be fuzzy uniformly continuous if for every $(N, \star) \in \mathcal{N}$ and $t > 0$ there exist $(M, *) \in \mathcal{M}$ and $s > 0$ such that $M(x, y, s) \leq N(f(x), f(y), t)$ for all $x, y \in X$.

PSUnif $\equiv$category of spaces endowed with a probabilistic uniform structure

Proposition

Let us consider the map $\mathfrak{S} : \mathsf{PUnif} \to \mathsf{PSUnif}$ *given by*

$$\mathfrak{S}((X, \mathcal{U}, *)) = (X, \mathfrak{s}(\mathcal{U}), *) = (X, \mathcal{M}_{\mathcal{U}}, *)$$

where $(\mathfrak{s}(\mathcal{U}), *) = (\mathcal{M}_{\mathcal{U}}, *)$ *is the probabilistic uniform structure of all fuzzy pseudometrics* $(M, *)$ *on* X *such that* $M(\cdot, \cdot, t) \in \mathcal{U}$ *for all* $t > 0$*, i. e.*

$$\mathcal{M}_{\mathcal{U}} = \{(M, *) \in \mathsf{FMet}(*) : \mathcal{U}_M \subseteq \mathcal{U}\}$$

and

$$\mathfrak{S}(f) = f$$

for every morphism f *in* PUnif*. Then* $\mathfrak{S}$ *is a covariant fully faithful functor.*

Proposition

Let us consider the map $\Upsilon : \mathsf{PSUnif} \to \mathsf{PUnif}$ *given by*

$$\Upsilon((X, \mathcal{M}, *)) = (X, \upsilon(\mathcal{M}), *) = (X, \mathcal{U}_{\mathcal{M}}, *)$$

where $(\mathcal{U}_{\mathcal{M}}, *)$ *is the probabilistic uniformity which has as base the family* $\{M(\cdot, \cdot, t) : t > 0, (M, *) \in \mathcal{M}\}$ *and*

$$\Upsilon(f) = f$$

for every morphism f *in* PSUnif. *Then* Υ *is a fully faithful covariant functor.*

Theorem

The following diagram commutes:

$$\begin{array}{ccc} \mathrm{Unif} & \underset{\Psi}{\overset{\Phi_*}{\rightleftarrows}} & \mathrm{FUnif}(*) \\ {\scriptstyle \omega_*}\downarrow & & \downarrow{\scriptstyle i} \\ \mathrm{PUnif}(*) & \underset{\mathfrak{S}}{\overset{\Upsilon}{\leftrightarrows}} & \mathrm{PSUnif}(*) \end{array}$$

where i denotes the inclusion functor.

Corollary

Lowen's functor ω_* can be factorized as follows:

$$\omega_* = \Upsilon \circ i \circ \Phi_*.$$

Theorem

The following diagram commutes:

$$\begin{array}{ccc}
\mathrm{PUnif}(*) & \underset{\Upsilon}{\overset{\mathfrak{S}}{\rightleftarrows}} & \mathrm{PSUnif}(*) \\
\Big\downarrow \mathcal{S} & & \Big\downarrow \mathcal{S}_s \\
\mathrm{Unif} & \underset{\Psi}{\overset{\Phi_*}{\rightleftarrows}} & \mathrm{FUnif}(*)
\end{array}$$

where

- $\mathcal{S}(\mathcal{U}) = \iota(\{\sup_{\varepsilon \in I_0}(F_\varepsilon - \varepsilon) : (F_\varepsilon)_{\varepsilon \in I_0} \in \mathcal{F}^{I_0}\})$;
- $\mathcal{S}_s(\mathcal{M}) = \{(N, *) \in \mathrm{FMet}(*) :$ *for all* $\varepsilon \in I_0$ *and* $t > 0$ *there exist* $\delta \in I_0, s > 0, (M, *) \in \mathcal{M}$ *such that* $M(x, y, s) > 1 - \delta$ *implies* $N(x, y, t) > 1 - \varepsilon\}$