Hereditarily indecomposable continua and entropy.

(joint work with Jan P. Boroński and Alex Clark)

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Indecomposable arc-like continua

1. continuum C is arc-like if for every $\varepsilon > 0$ there is an ε-map $\pi\colon C \to I$ (i.e. $\operatorname{diam} \pi^{-1}(x) < \varepsilon$ for every $x \in I$).
2. circle-like, tree-like, graph-like are defined analogously.
3. continuum C is indecomposable if is not the union of two proper subcontinua.
4. hereditarily indecomposable if all nondegenerate subcontinua are indecomposable.
5. arc-like hereditarily indecomposable continuum is topologically unique - we call it the pseudoarc (Knaster; Moise; Bing).

- (1951) R.H. Bing: pseudo-circle, a hereditarily indecomposable circle-like continuum that separates the plane into exactly two components. (uniqueness! Fearnley&Rogers)

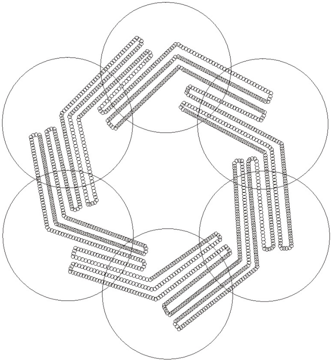


Figure: Construction by crooked circular chain (picture by Charatonik&Prajs&Pyrih)

- We call an arc A ϵ-*crooked* if for any pair of points p and q in A there exist points r and s between p and q such that r is between p and s, $|p - s| < \epsilon$, and $|r - q| < \epsilon$.

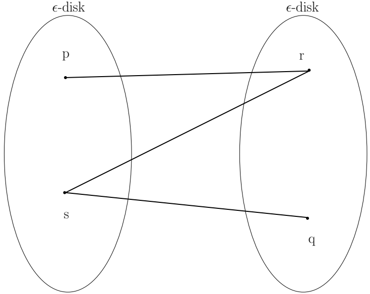
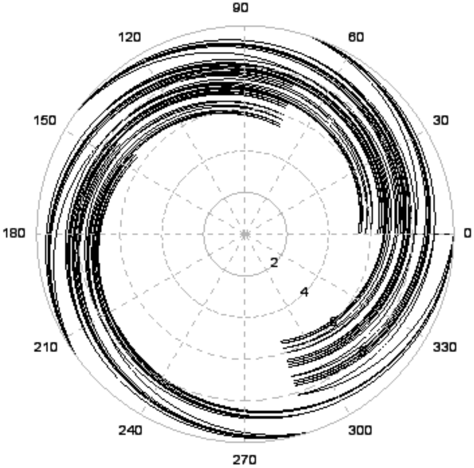


Figure: ϵ-crooked arc from p to q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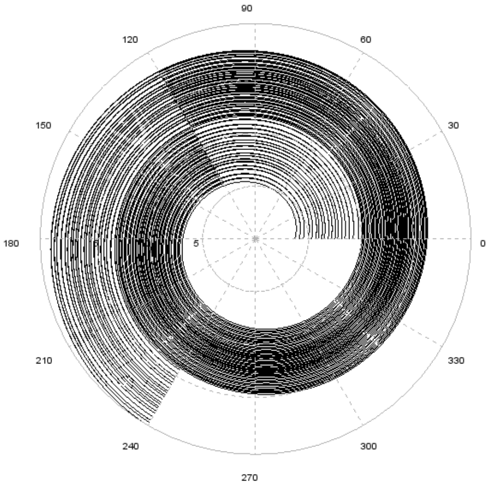

90
120
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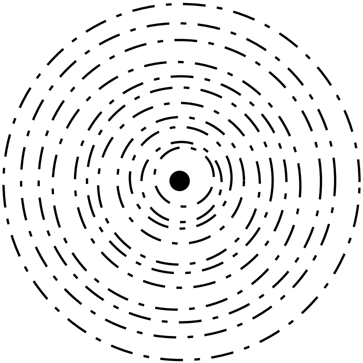
270

- (1960) Fearnley & Rogers: pseudo-circle is not homogeneous
- (1986) Kennedy & Rogers: pseudo-circle is uncountably nonhomogeneous
- (2011) Sturm: pseudo-circle is not continuously homogeneous

- (1991) Heath: pseudo-circle *n*-fold covers itself
- (1991) Bellamy & Lewis: the two-point compactification of the universal cover of the pseudo-circle is the pseudo-arc

M. Brown (1958): There exists a continuous decomposition of $\mathbb{R}^2 \setminus \{(0,0)\}$ into pseudo-circles.



- (1982) Handel: pseudo-circle as an attracting minimal set of a C^∞-smooth diffeomorphism of the plane
- (1996) Kennedy&Yorke: constructed a C^∞ diffeomorphism on a 7-manifold which has an invariant set with an uncontable number of pseudocircle components and is stable to C^1 perturbations
- (2010) Chéritat: pseudo-circle as a boundary of a Siegel disk of a holomorphic map

Inverse limits

1. in general, inverse limit -
$\varprojlim\{\{f_i\}_{i=0}^{\infty}, X\} = \{(x_0, x_1, \ldots) : x_i \in X, f_i(x_{i+1}) = x_i\}$

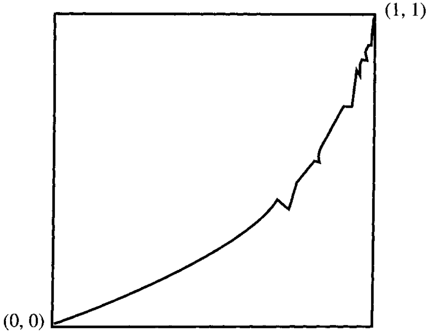
2. we are interested in cases when there is one bonding map:
$$\mathbb{X} = \varprojlim\{f, X\} = \{(x_0, x_1, \ldots) : x_i \in X, f(x_{i+1}) = x_i\}$$

3. shift homeo. - $\sigma_f(x_0, x_1, \ldots) = (f(x_0), x_0, x_1, \ldots)$

4. f and σ_f share many dynamical properties, e.g.
   - dense periodic points
   - admissible periods of periodic points
   - $h_{\text{top}}(f) = h_{\text{top}}(\sigma_f)$
   - .....

Pseudoarc

1. If $f \in C(I)$ has some special properties, then $\mathbb{X}$ is a pseudoarc.
2. Then we can study dynamical properties of the homeomorphism σ_f in terms of f.
3. Example of G. Henderson (Duke Math. J., 1964):



Method of Minc and Transue

Theorem (Kawamura, Tuncali & Tymchatyn)

Let G be a topological graph and $f\colon G \to G$ a piecewise linear surjection which satisfies the following condition (topological exactness):

- for each open subset U of G, there is a positive integer n such that $f^n(U) = G$.

Then for each $\varepsilon > 0$ there is a map $f_\varepsilon\colon G \to G$ which is ε-close to f such that (G, f_ε) is hereditarily indecomposable.

Theorem (Kościelniak & O. & Tuncali)

Let G be a topological graph and let $\mathcal{K}$ be a triangulation of G. For every topologically exact map $f\colon G \to G$ and every $\varepsilon > 0$ there is a topologically mixing map $f_\varepsilon\colon G \to G$ with the shadowing property, which is ε-close to f such that (G, f_ε) is hereditarily indecomposable and $f(x) = f_\varepsilon(x)$ for every vertex in $\mathcal{K}$.

Theorem (Barge & Martin)

Every continuum $\mathbb{X} = \varprojlim\{f, [0,1]\}$, can be embedded into a disk D in such a way that

(i) $\mathbb{X}$ is an attractor of a homeomorphism $h\colon D \to D$,

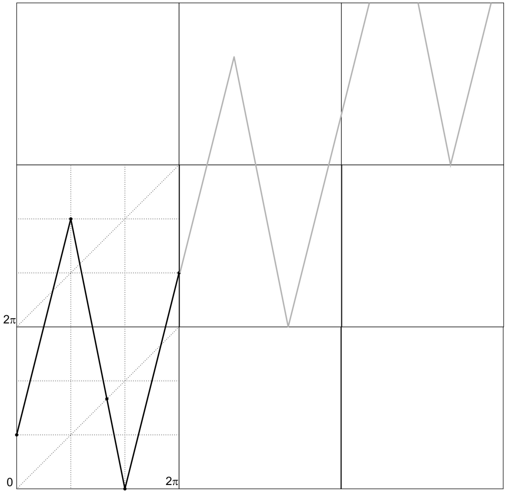
(ii) $h|_{\mathbb{X}} = \sigma_f$; i.e. h restricted to $\mathbb{X}$ agrees with the shift homeomorphism induced by f, and

(iii) h is the identity on the boundary of D.

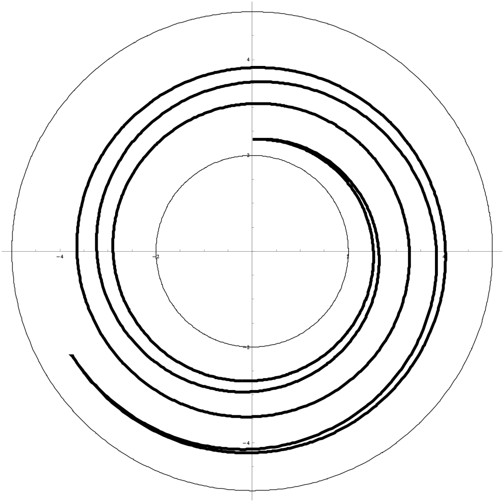
Remark

It was pointed by Barge & Roe that the same is true if f is a degree ± 1 circle map and h is an annulus homeomorphism.

Degree one map



Embedding



Pseudo-circle as strange attractor

Theorem (Barge&Gillette, 1991)

Suppose $h\colon A \to A$ is an orientation preserving annulus homeomorphism with an invariant cofrontier C. If the rotation number of $h|_C$ is not unique then C is indecomposable, the set of rotation numbers contains an interval, and each rational rotation number is realized by a periodic orbit.

Theorem (Boroński & O., 2015)

There exists a 2-torus homeomorphism h homotopic to identity such that:

- *a pseudocircle C is a strange attractor for h (i.e. set of rotation numbers on C contains an interval),*
- *$T|_C$ is topologically mixing and has positive entropy,*
- *the set of rotation numbers of $T|_C$ is an interval.*

An old question

Question 1 (Barge, 1989?)

Is every real number the entropy of some homeomorphism on the pseudo-arc?

Theorem (Mouron, 2012)

If $f \in C(I)$ is such that the inverse limit $\mathbb{X}$ is the pseudoarc then $h_{\text{top}}(f) \in \{0, \infty\}$.

- The answer to Barge's question is still unknown.
- With Example of Henderson + Minc and Transue technique we see that both cases 0, ∞ can be obtained in practice.

A related result

- We proved the following (with other methodology than Mouron).

Theorem (Boroński & O.)

If $f \in C(G)$ is such that the inverse limit $\mathbb{X}$ is the hereditarily indecomposable then:

1. $h_{\text{top}}(f) \in \{0, \infty\}$,

- it is known that there is a homeomorphism of the pseudo-circle with zero entropy - example of M. Handel.

On the circle

Theorem (Boroński & O.)

If $\mathbb{X} = \varprojlim(\mathbb{S}^1, f)$ is hereditarily indecomposable, and $\mathbb{X}$ is not the pseudo-arc (i.e. $\deg f \neq 0$) then $h_{top}(f) = \infty$

- Main ingredient is characterization of Auslander and Katznelson of periodic-point-free circle homeomorphisms,
- We believe that $h_{\mathrm{top}}(f) = \infty$ when G has no endpoints, but have no proof so far...

Corollary

Suppose $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a map with $\deg(f) = 1$. If $\Lambda_f = \varprojlim(\mathbb{S}^1, f)$ is hereditarily indecomposable then the rotation set $\rho(f)$ (after embedding by BBM) is nondegenerate.

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Theorem (Boroński & O.)

If $\mathbb{X} = \varprojlim(\mathbb{S}^1, f)$ *is hereditarily indecomposable, and* $\mathbb{X}$ *is not the pseudo-arc (i.e.* $\deg f \neq 0$*) then* $h_{top}(f) = \infty$

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M. Handel - Anosov-Katok type construction

- (1982) M. Handel: pseudo-circle as minimal set and attractor

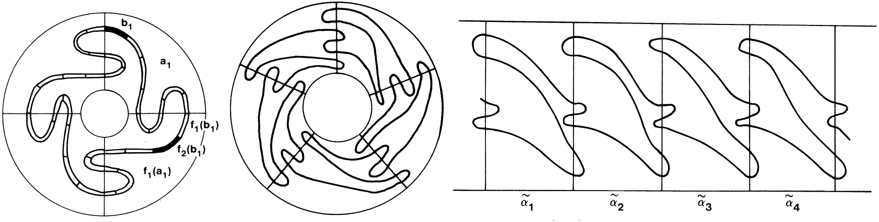


Figure: by M. Handel

Theorem (Handel, 1982)

There exists a C^∞-smooth diffeomorphism of the plane F with pseudo-circle as an attracting minimal set. In addition, F has a well defined irrational rotation number but is not semi-conjugate to a circle rotation (Thurston?).

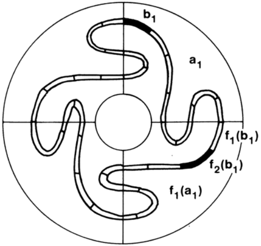
Question 2 (Auslander, AIMS Madrid 2014)

Does Handel's homeomorphism has any proximal pairs?

- A pair of points x, y is **proximal** for h if

$$\liminf_{n\to\infty} d(h^n(x), h^n(y)) = 0.$$

Dynamical properties of Handel's construction

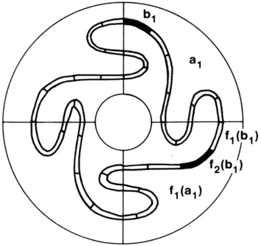


1. Handel's example is (implicitly contained in Handel's paper)
   - minimal
   - uniformly rigid

Observation (Boroński, Clark, O.)

Handel's example is weakly mixing (in particular, proximal pairs form a residual set).

Dynamical properties of Handel's construction



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Uniformly rigid, minimal & weak mixing

1. Examples of minimal, weakly mixing and uniformly rigid systems were first constructed in dimension 2 or higher by Glassner and Maon in 1989.
2. These examples include $\mathbb{T}^n$ for every $n \geq 2$.
3. Dynamical properties can be slightly extended beyond topological weak mining (e.g. weakly mixing measure)
4. or on other surfaces (e.g. Klein bottle; example by K. Yancey)
5. Up to our knowledge, this is the only second class of such examples, and the first in dimension 1.

Recall:

Question 1 (Barge, 1989)

Is every real number the entropy of some homeomorphism on the pseudo-arc?

Question 3 (Mouron 2009)

Does there exist a homeomorphism of a hereditarily indecomposable continuum with topological entropy other than 0 or ∞?

A solution...

Theorem (Boronski & Clark & O.)

There exist hereditarily indecomposable continua X satisfying:

1. *there is $\alpha \in (0,1)$ and a homeomorphism $T_\alpha : X \to X$ with $h_{top}(T_\alpha) = \alpha$ (in fact there are infinitely many such α)*
2. *X occurs as an invariant minimal set with intermediate complexity within an attractor of a smooth diffeomorphism F of a 4 dimensional manifold, and*
3. *the restriction $F|X$ is weakly mixing.*

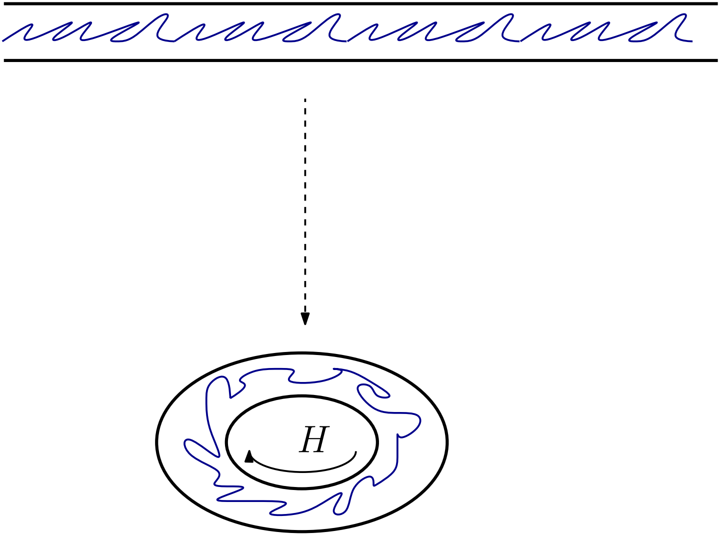
A brief look on formal methodology...

Theorem (Boronski & Clark & O.)

Assume that H is a HAK homeomorphism (e.g. the one from Handel's construction) with a nonzero rotation number α on pseudo-circle Ψ. Then $h_{top}(H_C) = |\alpha| h_{top}(h)$.

H_C "lives" in the quotient space $([0,1] \times \mathbb{R}) \times C / \approx$, where:

- $h\colon C \to C$ is a minimal homeomorphism,
- $((s,r),c) \approx ((s',r'),c')$
  if and only if
- $s = s'$ and there is an $n \in \mathbb{Z}$ satisfying $r' = r + n$ and $c' = h^{-n}(c)$,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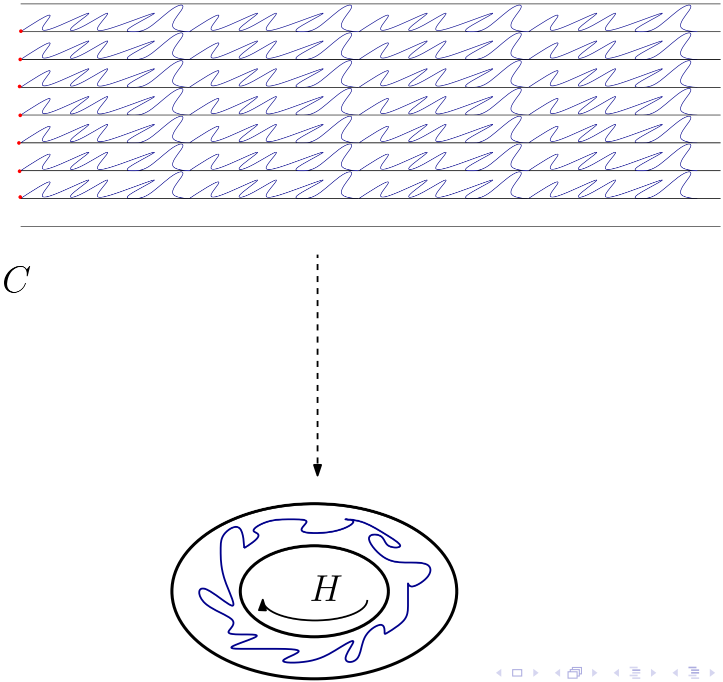
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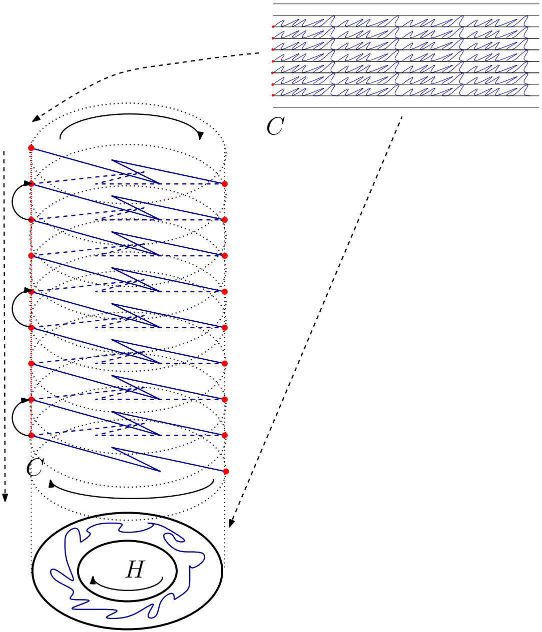
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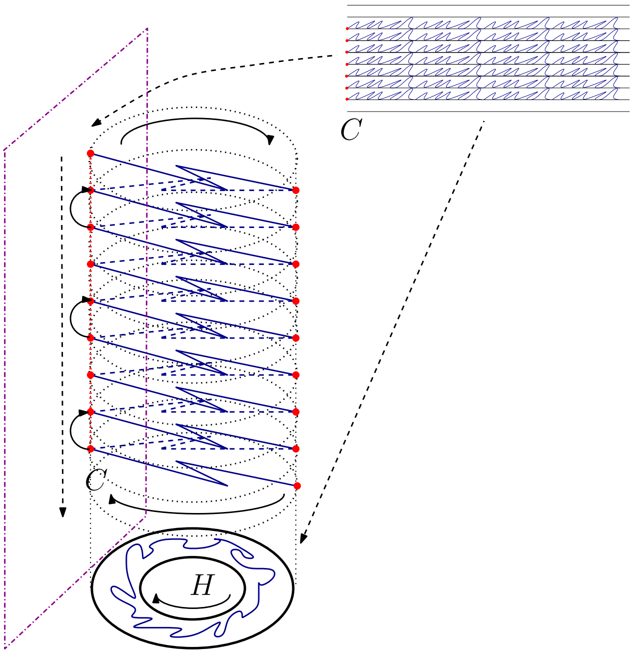
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Some further questions

Question 4a

Does for every $\alpha \in \mathbb{R}$ there exist a pseudo-circle homeomorphism with a well defined rotation number α?

Question 4b

Is there a hereditarily indecomposable continuum X such that for every $t \geq 0$ there is a homomorphism $F_t \colon X \to X$ of entropy $h_{\text{top}}(F_t) = t$?

Note: See a series of papers by John Mayer, and a paper by Mark Turpin.

Question 5

Is there an indecomposable cofrontier that admits a minimal homeomorphism semi-conjugated to an irrational circle rotation?

Theorem (Boroński & Clark & O.)

Let $\Psi \subset \mathbb{A}$ *be an essential pseudo-circle attracting all the points from* $\operatorname{int}\mathbb{A}$ *and assume that* $H\colon \mathbb{A} \to \mathbb{A}$ *is a homemomorphism with a nondegenerate rotation set. Then* $h_{top}(H|_\Psi) = h_{top}(H) = +\infty$.

Question 5

Let $\Psi \subset \mathbb{A}$ be an essential pseudo-circle and assume that $H\colon \mathbb{A} \to \mathbb{A}$ is a homemomorphism with a nondegenerate rotation set. Is it true that $h_{\text{top}}(H|_\Psi) = +\infty$.

That's all...

Thank you for your attention.

J. Boroński, A. Clark, P. Oprocha

New exotic minimal sets from pseudo-suspensions of Cantor systems

Preprint at arXiv:1609.09121.