Orbits of linear operators and Banach space geometry

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Let T be a bounded linear operator on a Banach space X. The study of its orbits is related to the invariant subset/subspace problem. For example, T does not have any invariant subset iff, for each $x \neq 0$, $O_T(x) := \{T^n x, n \geqslant 0\}$ is dense in X.

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Theorem (Read)

Such an operator exists on the space ℓ^1.

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Basic problem : Constructing "good" vectors x such that $\|T^n x\|$ is not too far away from $\|T^n\|$.

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Theorem (Muller, 2003)

Let T be a bounded linear operator on X, and let (a_n) be a sequence of non negative numbers such that $a_n \to 0$. Then, the set

$$\{x \in X, \|T^n x\| \geqslant a_n \|T^n\| \text{ for infinitely many n's}\}$$

is residual in X.

Definition

A subset E of a Banach space X is called porous if there exists $\lambda \in]0,1[$ such that the following is true: for every $x \in E$ and every $\epsilon > 0$, there exists a point $y \in X$ such that $0 < \|y - x\| < \epsilon$ and $E \cap B(y, \lambda\|x - y\|)$ is empty. A countable union of porous sets is called a σ-porous set.

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Definition

Let X be a separable Banach space. A set $E \subset X$ is said to be Haar-null if there exists a Borel probability measure m on X such that for every $x \in X$, the translate $x + E$ has m-measure 0.

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Theorem

Let X be a Banach space (real or complex) and (T_n) be a sequence of bounded linear operators on X. Let also (a_n) be a sequence of non-negative numbers such that $a_n \to 0$. Then, the set

$$\{x \in X, \|T_n x\| \geqslant a_n \|T_n\| \text{ for infinitely many n's}\}$$

has a complement which is σ-porous. If the space X is separable, then this complement is also Haar-null.

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Let X be a Banach space, $q < 1$, and $T \in \mathcal{L}(X)$ be a non-nilpotent operator, then the set

$$\left\{ x, \sum_{n=1}^{\infty} \left(\frac{\|T^n x\|}{\|T^n\|} \right)^q = \infty \right\}$$

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Remarque

There are examples showing that the constants 1 and 2 are optimal for Banach space and Hilbert space operators.

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Theorem (K. Ball, 1991, 2001)

Let $(f_n) \subset X^*$ such that $\|f_n\| = 1$ for each n. Let also $(\alpha_n) \subset \mathbb{R}^+$ such that $\sum_{n=1}^{\infty} \alpha_n < 1$. Then there is a point x, $\|x\| = 1$ such that for each n, $|\langle f_n, x\rangle| \geqslant \alpha_n$.

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The condition $\sum_{n=1}^{\infty} \alpha_n < 1$ can be improved by $\sum_{n=1}^{\infty} \alpha_n^2 < 1$ for complex Hilbert spaces.

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One can apply the Ball theorem to $((T^n)^* f_n / \|(T^n)^* f_n\|)_n$. There exists $x \in X$, $\|x\| = 1$ s.t for every n :

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so

$$\|T^n x\| \geqslant |\langle f_n, T^n x\rangle| = |\langle (T^n)^* f_n, x\rangle| \geqslant \alpha_n \|(T^n)^* f_n\| \geqslant \frac{\alpha_n}{2}\|T^n\|.$$

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Let X be Banach space. The modulus of asymptotic uniform smoothness of X is the fonction $\overline{\rho}_X(t)$ defined by

$$\overline{\rho}_X(t) = \sup_{\|x\|=1} \inf_{\dim(X/Y)<\infty} \sup_{y\in Y, \|y\|=1} (\|x+ty\|-1) \ (t \geqslant 0).$$

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⋆ $\overline{\rho}_{L^p(0,1)}(t) \asymp t^{\min(p,2)}$ when $t \to 0$ (Milman).

Theorem

Let X be a Banach space and $T \in \mathcal{L}(X)$ be a non nilpotent operator. Assume that: $\overline{\rho}_X(2t) = O(\overline{\rho}_X(t))$ when $t \to 0$. Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing map such that $\rho(t) > 0$ whenever $t > 0$ and

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Corollary

The best constants for ℓ^p, L^p *and* c_0 *are respectively* p, $\min(p, 2)$ *and* ∞.

1st case : there exists M of finite codimension s.t $\|T^n_{|M}\| \leqslant \frac{1}{2}\|T^n\|$ for infinitely many $n's$.

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There exists $(\alpha_i) \to 0$ s.t

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Find an increasing sequence (m_k) such that

$$\alpha_i \leqslant 2^{-k} \ (i \geqslant m_k) \text{ and } \prod_{i=m_k}^{\infty} (1 + \overline{\rho}_X(2^k \alpha_i)) \leqslant 2$$

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and choose $\beta_i > 0$ s.t $\prod_{i=1}^{\infty} (1 + \beta_i)$ converges.

Next, we will construct by induction 2 other sequences (n_j) and (u_i) such that: $n_1 < n_2 < \cdots$, $\|u_i\| = 1$. Further, these sequences will have to satisfy two properties. First, for each $l \geqslant 1$ and $j \leqslant l$:

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Secondly, for each $k \geqslant 1$, and $m_k \leqslant l \leqslant m_{k+1} - 1$:

$$\left\| \sum_{i=m_k}^{l} \alpha_i u_i \right\| \leqslant 2^{1-k} \left(\prod_{i=m_k}^{l} (1 + \beta_i) \right) \left(\prod_{i=m_k}^{l} (1 + \overline{\rho}_X(2^k \alpha_i)) \right). \tag{2}$$

Thank you for your attention!