

On the numerical radius of operators on Lebesgue spaces

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Notation

Basic notation

X Banach space.

- \mathbb{K} base field (it may be \mathbb{R} or \mathbb{C}),
- S_X unit sphere, B_X unit ball,
- X^* dual space,
- $L(X)$ bounded linear operators,
- $T^* \in L(X^*)$ adjoint operator of $T \in L(X)$.
- For $z \in \mathbb{K}$,
 - \bar{z} is the conjugate ($= z$ in the real case),
 - $\operatorname{Re} z$ is the real part ($= z$ in the real case).

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 - $\operatorname{Re} z$ is the real part ($= z$ in the real case).
- $L_p(\mu)$ (real or complex) Banach space of μ -measurable scalar functions with

$$\|x\|_p := \left(\int_{\Omega} |x|^p d\mu \right)^{\frac{1}{p}} < \infty$$

- $\ell_p^{(m)}$ m -dimensional L_p -space

Introduction

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Numerical index

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The 2000's results

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The new results

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The results

The results

The result for the absolute numerical radius

For $1 < p < \infty$, let $\kappa_p = p^{-\frac{1}{p}} q^{-\frac{1}{q}}$. Then

$$\sup \left\{ \int |x|^{p-1} |Tx| d\mu : x \in L_p(\mu), \|x\|_p = 1 \right\} \geq \kappa_p \|T\|$$

for every $T \in L(L_p(\mu))$ (real and complex cases).

- This inequality is best possible when $\dim(L_p(\mu)) \geq 2$.

The results

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Positivity of the numerical index

For $1 < p < \infty$, $p \neq 2$, there is a **positive** constant $n(L_p)$ such that

$$\sup \left\{ \left| \int |x|^{p-1} \operatorname{sign}(x) Tx \, d\mu \right| : x \in L_p(\mu), \|x\| = 1 \right\} \geq n(L_p) \|T\|$$

for every $T \in L(L_p(\mu))$ (real case).

- We do not know the best possibility for $n(L_p)$.

Schedule of the talk

- 1 Introduction
- 2 Numerical index of Banach spaces
- 3 The 2000's results on the numerical index on L_p -spaces
- 4 The new results on the numerical index of L_p -spaces

Numerical index of Banach spaces

2 Numerical index of Banach spaces

- Numerical range
- Numerical index: definition and basic properties
- Examples
- Stability properties
- Duality
- Renorming
- Open problems



F. F. Bonsall and J. Duncan

Numerical Ranges. Vol I and II.

London Math. Soc. Lecture Note Series, 1971 & 1973.



V. Kadets, M. Martín, and R. Payá.

Recent progress and open questions on the numerical index of Banach spaces.
RACSAM (2006)

Numerical range: Hilbert spaces

Hilbert space numerical range (Toeplitz, 1918)

- A $n \times n$ real or complex matrix

$$W(A) = \{(Ax | x) : x \in \mathbb{K}^n, (x | x) = 1\}.$$

- H real or complex Hilbert space, $T \in L(H)$,

$$W(T) = \{(Tx | x) : x \in H, \|x\| = 1\}.$$

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- H real or complex Hilbert space, $T \in L(H)$,

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Some properties

H Hilbert space, $T \in L(H)$:

- $W(T)$ is convex.
- In the complex case, $\overline{W(T)}$ contains the spectrum of T .
- If T is normal, then $\overline{W(T)} = \overline{\text{co}} \text{Sp}(T)$.

Numerical range: Banach spaces

Banach spaces numerical range (Bauer 1962; Lumer, 1961)

X Banach space, $T \in L(X)$,

$$V(T) = \{x^*(Tx) : x^* \in S_{X^*}, x \in S_X, x^*(x) = 1\}$$

Numerical range: Banach spaces

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Some properties

X Banach space, $T \in L(X)$:

- $V(T)$ is connected (not necessarily convex).
- In the complex case, $\overline{V(T)}$ contains the spectrum of T .
- In fact,

$$\overline{\text{co}}\text{Sp}(T) = \bigcap \overline{V(T)},$$

intersection taken over all numerical ranges $V(T)$ corresponding to equivalent norms on X .

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The new results
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Some motivations for the numerical range

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For Hilbert spaces

- It is a comfortable way to study the spectrum.
- It is useful to work with some concepts like hermitian operator, skew-hermitian operator, dissipative operator...
- It is useful to estimate spectral radii of small perturbations of matrices.

Some motivations for the numerical range

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For Banach spaces

- It allows to carry to the general case the concepts of hermitian operator, skew-hermitian operator, dissipative operators...
- It gives a description of the Lie algebra corresponding to the Lie group of all onto isometries on the space.
- It gives an easy and quantitative proof of the fact that Id is an strongly extreme point of $B_{L(X)}$ (MLUR point).

Numerical index of Banach spaces: definitions

Numerical radius

X Banach space, $T \in L(X)$. The **numerical radius** of T is

$$v(T) = \sup \left\{ |x^*(Tx)| : x^* \in S_{X^*}, x \in S_X, x^*(x) = 1 \right\}$$

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Remark

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Exponential formula

$\|\exp(\rho T)\| \leq e^{|\rho|v(T)}$ ($\rho \in \mathbb{K}$) and $v(T)$ is best possible.

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Numerical index (Lumer, 1968)

X Banach space, the **numerical index** of X is

$$\begin{aligned} n(X) &= \inf \{ v(T) : T \in L(X), \|T\| = 1 \} \\ &= \max \{ k \geq 0 : k\|T\| \leq v(T) \quad \forall T \in L(X) \} \end{aligned}$$

Numerical index of Banach spaces: basic properties

Some basic properties

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- $n(X) = 1$ iff v and $\|\cdot\|$ coincide.
- $n(X) = 0$ iff v is not an equivalent norm in $L(X)$

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- X complex $\Rightarrow n(X) \geq 1/e$.
(Bohnenblust–Karlin, 1955; Glickfeld, 1970)

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Some basic properties

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- X complex $\Rightarrow n(X) \geq 1/e$.

(Bohnenblust–Karlin, 1955; Glickfeld, 1970)

- Actually,

$$\{n(X) : X \text{ complex, } \dim(X) = 2\} = [e^{-1}, 1]$$

$$\{n(X) : X \text{ real, } \dim(X) = 2\} = [0, 1]$$

(Duncan–McGregor–Pryce–White, 1970)

Numerical index of Banach spaces: some examples

Examples

- ① H Hilbert space, $\dim(H) > 1$,

$$\begin{aligned} n(H) &= 0 && \text{if } H \text{ is real} \\ n(H) &= 1/2 && \text{if } H \text{ is complex} \end{aligned}$$

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- ② $n(L_1(\mu)) = 1$ μ positive measure
 $n(C(K)) = 1$ K compact Hausdorff space
(Duncan et al., 1970)

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$$n(C(K)) = 1 \quad K \text{ compact Hausdorff space}$$

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- ③ If A is a C^* -algebra $\Rightarrow \begin{cases} n(A) = 1 & A \text{ commutative} \\ n(A) = 1/2 & A \text{ not commutative} \end{cases}$

(Huruya, 1977; Kaidi–Morales–Rodríguez, 2000)

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- ④ If A is a function algebra $\Rightarrow n(A) = 1$

(Werner, 1997)

Numerical index of Banach spaces: some examples II

More examples

- ⑤ For $n \geq 2$, the unit ball of X_n is a $2n$ regular polygon:

$$n(X_n) = \begin{cases} \tan\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is even,} \\ \sin\left(\frac{\pi}{2n}\right) & \text{if } n \text{ is odd.} \end{cases}$$

(M.-Merí, 2007)

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- ⑥ Every finite-codimensional subspace of $C[0, 1]$ has numerical index 1
(Boyko–Kadets–M.–Werner, 2007)

Stability properties

Direct sums of Banach spaces (M.-Payá, 2000)

$$n\left(\left[\bigoplus_{\lambda \in \Lambda} X_\lambda\right]_{c_0}\right) = n\left(\left[\bigoplus_{\lambda \in \Lambda} X_\lambda\right]_{\ell_1}\right) = n\left(\left[\bigoplus_{\lambda \in \Lambda} X_\lambda\right]_{\ell_\infty}\right) = \inf_{\lambda} n(X_\lambda)$$

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Vector-valued function spaces (López–M.–Merí–Payá–Villena, 200's)

E Banach space, μ positive measure, K compact space. Then

$$n(C(K, E)) = n(C_w(K, E)) = n(L_1(\mu, E)) = n(L_\infty(\mu, E)) = n(E),$$

and $n(C_{w^*}(K, E^*)) \leq n(E)$

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Tensor products (Lima, 1980)

There is no general formula neither for $n(X \widetilde{\otimes}_\varepsilon Y)$ nor for $n(X \widetilde{\otimes}_\pi Y)$:

- $n(\ell_1^{(4)} \widetilde{\otimes}_\pi \ell_1^{(4)}) = n(\ell_\infty^{(4)} \widetilde{\otimes}_\varepsilon \ell_\infty^{(4)}) = 1.$
- $n(\ell_1^{(4)} \widetilde{\otimes}_\varepsilon \ell_1^{(4)}) = n(\ell_\infty^{(4)} \widetilde{\otimes}_\pi \ell_\infty^{(4)}) < 1.$

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Numerical index and duality

Numerical index and duality

Proposition

X Banach space, $T \in L(X)$. Then

$$\bullet \sup \operatorname{Re} V(T) = \lim_{\alpha \rightarrow 0^+} \frac{\|\operatorname{Id} + \alpha T\| - 1}{\alpha}.$$

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Question (From the 1970's)

Is $n(X) = n(X^*)$?

Negative answer (Boyko–Kadets–M.–Werner, 2007)

Consider the space

$$X = \{(x, y, z) \in c \oplus_{\infty} c \oplus_{\infty} c : \lim x + \lim y + \lim z = 0\}.$$

Then, $n(X) = 1$ but $n(X^*) < 1$.

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Numerical index and renorming

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Proposition (Finet–M.–Payá, 2003)

X separable or reflexive, then

$$\{n(Y) : Y \simeq X\} \supseteq \begin{cases} [0, 1[& \text{real case} \\ [e^{-1}, 1[& \text{complex case} \end{cases}$$

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Theorem (Avilés–Kadets–M.–Merí–Shepelska, 2010)

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Proposition (Boyko–Kadets–M.–Merí, 2009)

X separable, $X \supset c_0 \implies \exists Y \simeq X$ with $n(Y) = 1$.

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Some interesting open problems

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- ① Characterize (without operators) Banach spaces with numerical index 1.

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- ② X (real) with $n(X) = 1$, $X \supset c_0$ or $X \supset \ell_1$?

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- ④ For instance, if $X^* \supset \ell_1$ can X be renormed with numerical index 1 ?
- ⑤ Find isomorphic or isometric conditions assuring that $n(X) = n(X^*)$.

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- ③ Characterize those X admitting a renorming with numerical index 1.
- ④ For instance, if $X^* \supset \ell_1$ can X be renormed with numerical index 1 ?
- ⑤ Find isomorphic or isometric conditions assuring that $n(X) = n(X^*)$.

The oldest open problem

Calculate the numerical index of “classical” spaces.

- In particular, calculate $n(L_p(\mu))$.

The 2000's results on the numerical index on L_p -spaces

③ The 2000's results on the numerical index on L_p -spaces



E. Ed-dari.

On the numerical index of Banach spaces.
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M. Martín, and J. Merí.

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Numerical index of L_P -spaces

Known results on the numerical index of L_p -spaces

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Known results on the numerical index of L_p -spaces

- ① $n(\ell_p) \leq n(\ell_p^{(m+1)}) \leq n(\ell_p^{(m)})$ for $m \in \mathbb{N}$.
(M.-Payá, 2000)

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(M.-Payá, 2000)

② $n(L_p[0, 1]) = n(\ell_p) = \lim_{m \rightarrow \infty} n(\ell_p^{(m)}) = \inf_{m \in \mathbb{N}} n(\ell_p^{(m)})$.

(Ed-Dari, 2005 & Ed-Dari-Khamsi, 2006)

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(Ed-Dari, 2005 & Ed-Dari-Khamisi, 2006)

③ In the real case,

$$\max \left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leq n(\ell_p^{(2)}) \leq v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}$

(M.-Merí, 2009)

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Ideas behind the proofs I

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The numerical index decreases with the dimension

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Proposition (M.-Payá, 2000)

$Z = U \oplus V$ with absolute sum (i.e. $\|u+v\| = f(\|u\|, \|v\|)$ for $u \in U, v \in V$).
 $\implies n(Z) \leq \min\{n(U), n(V)\}$.

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$$n(\ell_p) \leq n(\ell_p^{(m+1)}) \leq n(\ell_p^{(m)}) \text{ for } m \in \mathbb{N}.$$

Proposition (M.-Payá, 2000)

$Z = U \oplus V$ with absolute sum (i.e. $\|u+v\| = f(\|u\|, \|v\|)$ for $u \in U, v \in V$).
 $\implies n(Z) \leq \min\{n(U), n(V)\}$.

Proof of the decreasing

- $\ell_p^{(m)}$ is an absolute summand in both $\ell_p^{(m+1)}$ and in ℓ_p .

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Ideas behind the proofs II

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One inequality

$$n(L_p[0,1]) \leq \lim_{m \rightarrow \infty} n(\ell_p^{(m)}).$$

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Proposition (M.–Merí–Popov–Randrianantoanina, 2011)

E order continuous Köthe space, X Banach space

$$\implies n(E(X)) \leq n(X).$$

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Proof of the inequality

- $E = L_p[0,1]$, $X = \ell_p^{(m)}$.
- $E \equiv E(X)$ so $n(E) \leq n(\ell_p^{(m)})$.

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Ideas behind the proofs III

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The reversed inequality

$$n(L_p[0, 1]) \geq \lim_{m \rightarrow \infty} n(\ell_p^{(m)}) \quad \text{and} \quad n(\ell_p) \geq \lim_{m \rightarrow \infty} n(\ell_p^{(m)}).$$

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Proposition (M.-Merí-Popov-Randrianantoanina, 2011)

Z Banach space, $\{Z_i\}_{i \in I}$ increasing family of one-complemented subspaces whose union is dense. Then, $\implies n(Z) \geq \limsup_{i \in I} n(Z_i)$.

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Corollary

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Proof of the inequality

- $Z = \ell_p$, (e_m) canonical basis $\implies Z_m \equiv \ell_p^{(m)}$ for all $m \in \mathbb{N}$.
- $E = L_p[0,1]$, (e_m) Haar system $\implies Z_m \equiv \ell_p^{(m)}$ for $m = 2^k$ ($k \in \mathbb{N}$).

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Ideas behind the proofs IV

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The two-dimensional case

In the real case,

$$\max \left\{ \frac{1}{2^{1/p}}, \frac{1}{2^{1/q}} \right\} M_p \leq n(\ell_p^{(2)}) \leq M_p \quad \text{where} \quad M_p = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1+t^p}$$

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Proposition (Duncan-McGregor-Pryce-White, 1970)

$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ operator in $\ell_p^{(2)}$. Then

$$v(T) = \max \left\{ \max_{t \in [0,1]} \frac{|a + dt^p| + |bt + ct^{p-1}|}{1+t^p}, \max_{t \in [0,1]} \frac{|d + at^p| + |ct + bt^{p-1}|}{1+t^p} \right\}.$$

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Proof of the result

- $n(\ell_p^{(2)}) \leq M_p$ since $\left\| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\| = 1$ and $v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = M_p$.

Ideas behind the proofs IV

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- We compare $v(T)$ with M_p , but we use $\|T\|_1$ and $\|T\|_\infty$ instead of $\|T\|_p$.

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Questions

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The 2010's results

- We leaved the finite-dimensional approach and introduce the **absolute numerical radius**.

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The 2010's results

- We leaved the finite-dimensional approach and introduce the **absolute numerical radius**.
- This allows to show that $n(L_p[0, 1]) > 0$ in the real case.

The new results on the numerical index of L_p -spaces

4 The new results on the numerical index of L_p -spaces



M. Martín, J. Merí, M. Popov.

On the numerical index of real $L_p(\mu)$ -spaces.
Isr. J. Math. (2011)



M. Martín, J. Merí, M. Popov.

On the numerical radius of operators in Lebesgue spaces.
Preprint

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The absolute numerical radius in L_p

The absolute numerical radius in L_p

The numerical radius in L_p

- For $x \in L_p(\mu)$, write $x^\# = |x|^{p-1} \operatorname{sign}(\bar{x})$.
- It is the unique element in $L_q(\mu)$ such that

$$\|x\|_p^p = \|x^\#\|_q^q \quad \text{and} \quad \int x x^\# d\mu = \|x\|_p \|x^\#\|_q = \|x\|_p^p.$$

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- Therefore, for $T \in L(L_p(\mu))$ one has

$$\begin{aligned} v(T) &= \sup \left\{ \left| \int x^\# T x d\mu \right| : x \in L_p(\mu), \|x\|_p = 1 \right\} \\ &= \sup \left\{ \left| \int |x|^{p-1} \operatorname{sign}(x) T x d\mu \right| : x \in L_p(\mu), \|x\|_p = 1 \right\} \end{aligned}$$

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Absolute numerical radius

For $T \in L(L_p(\mu))$,

$$\begin{aligned} |v|(T) &:= \sup \left\{ \int |x^\# T x| d\mu : x \in L_p(\mu), \|x\|_p = 1 \right\} \\ &= \sup \left\{ \int |x|^{p-1} |T x| d\mu : x \in L_p(\mu), \|x\|_p = 1 \right\}. \end{aligned}$$

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The absolute numerical index of L_p

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Obvious remark

$$v(T) \leq |v|(T) \leq \|T\| \text{ for every } T \in L(L_p(\mu)).$$

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Absolute numerical index

$$\begin{aligned} |n|(L_p(\mu)) &= \inf \left\{ |v|(T) : T \in L(L_p(\mu)), \|T\| = 1 \right\} \\ &= \max \left\{ k \geq 0 : k\|T\| \leq |v|(T) \quad \forall T \in L(L_p(\mu)) \right\}. \end{aligned}$$

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The first results

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Proposition 1

Write $M_p = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1+t^p}$. Then, in the real case,

$$v(T) \geq \frac{M_p}{6} |v|(T) \quad (T \in L(L_p(\mu))).$$

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Proposition 2

In the real case,

$$|v|(T) \geq \frac{1}{2} v(T_{\mathbb{C}}) \geq \frac{n(L_p^{\mathbb{C}}(\mu))}{2} \|T\| \quad (T \in L(L_p(\mu))).$$

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We do not know the value of $n(L_p^{\mathbb{C}}(\mu))$, but $n(X) \geq 1/e$ for complex spaces, so

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Theorem

In the real case, $n(L_p(\mu)) \geq \frac{M_p}{12e} > 0$ for $1 < p < \infty$, $p \neq 2$.

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We improved Proposition 2 calculating $|n|(L_p(\mu))$ for real and complex spaces

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Calculating $|n|(L_p(\mu))$ |

Calculating $|n|(L_p(\mu))$ |

The constant

$$\text{Set } \kappa_p := \max_{\tau > 0} \frac{\tau^{p-1}}{1 + \tau^p} = \max_{\lambda \in [0,1]} \lambda^{\frac{1}{q}} (1 - \lambda)^{\frac{1}{p}} = \frac{1}{p^{1/p} q^{1/q}}.$$

Calculating $|n|(L_p(\mu))$ |

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The best possibility for $|n|(L_p(\mu))$

If $\dim(L_p(\mu)) \geq 2$, then there is a (positive) operator $T \in L(L_p(\mu))$ with

$$\|T\| = 1, \quad |v|(T) = \kappa_p.$$

Calculating $|n|(L_p(\mu))$ |

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Calculating $|n|(L_p(\mu))$ |

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- For ℓ_p : consider the extension by zero of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Calculating $|n|(L_p(\mu))$ |

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- For ℓ_p : consider the extension by zero of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
- For $L_p[0,1]$:

$$Tf = 2 \left[\int_0^{1/2} f(s) ds \right] \chi_{[\frac{1}{2},1]} \quad (f \in L_p[0,1]).$$

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Calculating $|n|(L_p(\mu))$ II

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Theorem

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- Fix $T \in L(L_p(\mu))$ positive with $\|T\| = 1$, $\tau > 0$ and $\varepsilon > 0$.
- Find $x \geq 0$ with $\|Tx\|^p > 1 - \varepsilon$, set

$$y = x \vee \tau Tx \quad \text{and} \quad A = \{\omega \in \Omega : x(\omega) \geq \tau(Tx)(\omega)\},$$

and observe that

$$\|y\|^p = \int_A x^p d\mu + \int_{\Omega \setminus A} (\tau Tx)^p d\mu \leq 1 + \tau^p \quad \text{and} \quad y^\# = x^{p-1} \vee (\tau Tx)^{p-1}.$$

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- Taking supremum on $\tau > 0$ and $\varepsilon > 0$, we get $v(T) \geq \kappa_p$.

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One consequence and further results

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- 2 Numerical index of Banach spaces
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