Fractional Difference Equations and Applications

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$f : \mathbb{N}_0 \to X,$

$$\Delta f(n) := f(n+1) - f(n), \quad n \in \mathbb{N}_0$$

$$\Delta^m f(n) = \Delta^{m-1} \circ \Delta f(n), \quad m = 2, 3, \ldots$$

$$\Delta^{1/2} f(n) = \quad ?$$

- In 1956, Kutter mentioned by the first time differences of fractional order.
- In 1974, Diaz and Osler introduced a discrete fractional difference operator defined as an infinite series.
- In 1988, Grey and Zhang developed a fractional calculus for the discrete nabla (backward) difference operator.
- At the same time, Miller and Ross defined a fractional sum via the solution of a linear difference equation.
- in 2007, Atici and Eloe introduced the Riemann-Liouville like fractional difference by using the definition of fractional sum of Miller and Ross.

- In 2010, Anastassiou defined the Caputo like fractional difference by using also the notion of fractional sum from Miller and Ross.
- At the same year, Ferreira introduced the concept of left and right fractional sum/difference and started a fractional discrete-time theory of the calculus of variations.
- In 2011, Holm further developed and applied the tools of discrete fractional calculus to the arena of fractional difference equations.

Holm corrected some inconsistencies which were presented in previous papers, mainly concerning the domain of the involved functions in the calculus. Indeed, such inconsistencies were also noted by others authors before.

We observe that different reasonable definitions for fractional differences are not, in general, equivalent. From the point of view of analysis of boundary value problems, one of the more useful notions until now corresponds to those given by Atici and Eloe. Indeed, there is a good number of papers dealing with discrete fractional boundary value problems using their definition.

A problem that naturally arises, when one intends to study the subject of fractional difference equations, is the correct use of the definition of fractional difference. The reason seems to be in the own definition of fractional sum introduced by Miller and Ross. As remarked by Holm, the definition involves different domains of the functions, which produces confusion and inconsistencies in the applications to fractional difference equations. Moreover, difficulties of notation, make the statement of some results unwieldy.

The main purpose of this talk is to point out that some of the different definitions of fractional difference operators existing in the literature are related by simple translation. This will be done by a correct interpretation of the operators of fractional sums and fractional differences involved.

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- We compare three definitions of discrete fractional sums, due to Miller and Ross, Grey and Zhang, and Atici and Eloe. We relate all of them by translation.
- We show in a unified way the corresponding notions of fractional difference in the sense of Riemann-Lioville and Caputo.

Introduction

- We introduce a definition of Mittag-Leffler sequence and show how it helps to solve the fractional difference equation
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- We completely solve the problem

$$\Delta^{\alpha} u(n) = c, \quad n \in \mathbb{N}_0,$$

with both, initial and boundary initial values.

Introduction

- We solve the Gompertz fractional difference equation

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- We solve the Gompertz fractional difference equation

$$\Delta^\alpha u(n) = (b-1)u(n) + a, \quad n \in \mathbb{N}_0.$$

- We characterize the solutions of the nonlinear difference equation

$$\Delta^\alpha u(n) = f(n, u(n)), \quad n \in \mathbb{N}_0,$$

in terms of a Volterra difference equation.

For a real number a, we denote

$$\mathbb{N}_a := \{a, a+1, a+2, \dots\}.$$

and we write $\mathbb{N}_1 \equiv \mathbb{N}$. Let X be a complex Banach space. We denote by $s(\mathbb{N}_a; X)$ the vectorial space consisting of all vector-valued sequences $f : \mathbb{N}_a \to X$.
The forward Euler operator $\Delta_a : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ is defined by

$$\Delta_a f(t) := f(t+1) - f(t), \quad t \in \mathbb{N}_a.$$

For $m \in \mathbb{N}_2$, we define recursively $\Delta_a^m : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ by

$$\Delta_a^m := \Delta_a^{m-1} \circ \Delta_a,$$

and is called the m-th order forward difference operator.

We also denote $\Delta_a^0 \equiv I_a$, where $I_a : s(\mathbb{N}_a; X) \to s(\mathbb{N}_a; X)$ is the identity operator and $\Delta \equiv \Delta_0^1$. We define the translation (by $a \in \mathbb{R}$) operator $\tau_a : s(\mathbb{N}_a; X) \to s(\mathbb{N}_0; X)$ by

$$\tau_a g(n) := g(a+n), \quad n \in \mathbb{N}_0.$$

Note that $\tau_a^{-1} = \tau_{-a}$ and $\tau_{a+b} = \tau_a \circ \tau_b = \tau_b \circ \tau_a$. Moreover,

$$\Delta_a^m \circ \tau_a^{-1} = \tau_a^{-1} \circ \Delta_0^m.$$

In other words, the following diagram is commutative

$$\begin{array}{ccc} s(\mathbb{N}_a; X) & \overset{\Delta_a^m}{\longrightarrow} & s(\mathbb{N}_a; X) \\ \uparrow \tau_a^{-1} & & \uparrow \tau_a^{-1} \\ s(\mathbb{N}_0; X) & \underset{\Delta_0^m}{\longrightarrow} & s(\mathbb{N}_0; X). \end{array}$$

We define

$$k^{\alpha}(j) = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(j+1)}, \quad j \in \mathbb{N}_0. \tag{0.1}$$

For example, we have $k^1(j) = 1$ and $k^2(j) = j + 1$ for all $j \in \mathbb{N}_0$. More generally, for each $m \in \mathbb{N}$, we have

$$k^m(j) = \frac{(j+m-1)!}{(m-1)!j!}, \quad j \in \mathbb{N}_0.$$

An interesting case of (0.1) is the following:

$$k^{1/2}(j) = \frac{(2j)!}{4^j (j!)^2}, \quad j \in \mathbb{N}_0.$$

We now recall the following notions of fractional sum. We start by presenting the definition given by Miller and Ross (1989).

Definition

Let $\alpha > 0$ and $a \in \mathbb{R}$. The α-th fractional sum of a function f is defined by

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s), \tag{0.2}$$

where $t \in \mathbb{N}_{a+\alpha}$ and $t^{(\alpha)} := \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}$.

Remark

$t^{(\alpha)}$ is sometimes denoted by $t^{\underline{\alpha}}$.

The following definition was proposed by Atici and Eloe in 2009.

Definition

Let $\alpha > 0$. For any given positive real number a, the α-th fractional sum of a function f is

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t-s+1)^{\overline{\alpha-1}} f(s),$$

where $t \in \mathbb{N}_a$ and $t^{\overline{\alpha}} := \dfrac{\Gamma(t+\alpha)}{\Gamma(t)}$.

The following definition of the (backward nabla) discrete fractional sum was proposed by Grey and Zhang in 1988.

Definition

The (nabla) left fractional sum of order $\alpha > 0$ (starting from a) is defined by

$$_{GZ}\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-s+1)^{\overline{\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+1}$.

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We introduce

Definition

Let $f : \mathbb{N}_0 \to X$ be a vector-valued sequence and $\alpha > 0$. The $\alpha-$*th fractional sum* of f, denoted by $\Delta^{-\alpha} f$, is a vector valued sequence defined by means of the formula

$$\Delta^{-\alpha} f(n) := \sum_{j=0}^{n} k^{\alpha}(n-j) f(j), \quad n \in \mathbb{N}_0,$$

where

$$k^{\alpha}(j) = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(j+1)}, \quad j \in \mathbb{N}_0.$$

The function $\Delta^{-\alpha} : s(\mathbb{N}_0; X) \to s(\mathbb{N}_0; X)$ is called the fractional delta operator.

We note that

Remark

$\Delta^{-\alpha} \equiv \nabla_0^{-\alpha}$.

One of the reasons to choose this operator is because their flexibility to be handled by means of Z-transform methods. Moreover, it has a better behavior for mathematical analysis when we ask, for example, for definitions of fractional sums and differences on subspaces of $s(\mathbb{N}_0; X)$ like e.g. l_p spaces.

An unified approach

$$
\begin{array}{cccc}
s(\mathbb{N}_a;X) & \xrightarrow{\Delta_a^{-\alpha}} & s(\mathbb{N}_{a+\alpha};X) & \text{Miller-Ross} \\
\downarrow \tau_a & & \downarrow \tau_{a+\alpha} & \\
s(\mathbb{N}_0;X) & \xrightarrow{\Delta^{-\alpha}} & s(\mathbb{N}_0;X) & \text{Definition 4} \\
\uparrow \tau_a & & \uparrow \tau_a & \\
s(\mathbb{N}_a;X) & \xrightarrow{\nabla_a^{-\alpha}} & s(\mathbb{N}_a;X) & \text{Atici-Eloe} \\
\uparrow \tau_1 & & \uparrow \tau_1 & \\
s(\mathbb{N}_{a+1};X) & \xrightarrow[{}_{GZ}\nabla_a^{-\alpha}]{} & s(\mathbb{N}_{a+1};X) & \text{Grey-Zhang}
\end{array}
$$

We summarize the relations above in the next result.

Theorem

For all $\alpha > 0$ *and* $a \in \mathbb{R}$, *the following assertions hold:*

(i) $\tau_{a+\alpha} \circ \Delta_a^{-\alpha} = \Delta^{-\alpha} \circ \tau_a$.

(ii) $\tau_a \circ \nabla_a^{-\alpha} = \Delta^{-\alpha} \circ \tau_a$.

(iii) $\tau_{a+1} \circ {}_{GZ}\nabla_a^{-\alpha} = \Delta^{-\alpha} \circ \tau_{a+1}$.

Now, we present the definition of fractional differences. The next concept, given by Miller and Ross in 1989, is analogous to the definition of a fractional derivative in the sense of Riemann-Liouville.

Definition

Let $\alpha > 0$. The *α-th fractional Riemann-Liouville like difference* is defined by

$$\Delta_a^{\alpha} f(t) = \Delta_{a+(m-\alpha)}^{m} (\Delta_a^{-(m-\alpha)} f)(t), \quad t \in \mathbb{N}_{a+(m-\alpha)},$$

where $m-1 < \alpha < m$, $m = \lceil \alpha \rceil$. Here $\lceil \cdot \rceil$ denotes the ceiling number i.e. the smallest integer not less than α.

Analogously, following the sense of Riemann-Liouville described above, the next definition appears given by Abdeljawad and Atici in 2012.

Definition

The nabla (left) fractional difference operator of order $\alpha > 0$ is defined by

$$_{GZ}\nabla_a^{\alpha} f(t) = \Delta_{a+1}^{m}({}_{GZ}\nabla_a^{-(m-\alpha)} f)(t), \quad t \in \mathbb{N}_{a+1}$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

For completeness, we also may propose the following definition.

Definition

The nabla fractional difference operator of order $\alpha > 0$ is defined by

$$\nabla_a^{\alpha} f(t) = \Delta_a^m (\nabla_a^{-(m-\alpha)} f)(t), \quad t \in \mathbb{N}_a$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

An unified approach

Below we illustrate, by a commutative diagram, the relationship between the different operators acting in the definitions of the operators of fractional differences (in the sense of Riemann-Liouville).

$$
\begin{tikzcd}
s(\mathbb{N}_a; X) \arrow[r, "\Delta_a^{-(m-\alpha)}"] \arrow[d, "\tau_a"] & s(\mathbb{N}_{a+(m-\alpha)}; X) \arrow[r, "\Delta^m_{a+(m-\alpha)}"] \arrow[d, "\tau_{a+(m-\alpha)}"] & s(\mathbb{N}_{a+(m-\alpha)}; X \arrow[d, "\tau_{a+(m-\alpha)}"] \\
s(\mathbb{N}_0; X) \arrow[r, "\Delta^{-(m-\alpha)}"] & s(\mathbb{N}_0; X) \arrow[r, "\Delta_0^m"] & s(\mathbb{N}_0; X) \\
s(\mathbb{N}_a; X) \arrow[u, "\tau_a"] \arrow[r, "\nabla_a^{-(m-\alpha)}"] & s(\mathbb{N}_a; X) \arrow[u, "\tau_a"] \arrow[r, "\Delta_a^m"] & s(\mathbb{N}_a; X) \arrow[u, "\tau_a"] \\
s(\mathbb{N}_{a+1}; X) \arrow[u, "\tau_1"] \arrow[r, "{}_{GZ}\nabla_a^{-\alpha}"'] & s(\mathbb{N}_{a+1}; X) \arrow[u, "\tau_1"] \arrow[r, "\Delta^m_{a+1}"] & s(\mathbb{N}_{a+1}; X) \arrow[u, "\tau_1"]
\end{tikzcd}
$$

In the following diagram, we again emphasize the role of our main operator $\nabla_0^{\alpha} \equiv \Delta^{\alpha}$.

$$
\begin{array}{ccc}
s(\mathbb{N}_a; X) & \overset{\Delta_a^{\alpha}}{\rightarrow} & s(\mathbb{N}_{a+(m-\alpha)}; X) \\
\downarrow \tau_a & & \downarrow \tau_{a+(m-\alpha)} \\
s(\mathbb{N}_0; X) & \overset{\Delta^{\alpha}}{\rightarrow} & s(\mathbb{N}_0; X) \\
\uparrow \tau_a & & \uparrow \tau_a \\
s(\mathbb{N}_a; X) & \overset{\nabla_a^{\alpha}}{\rightarrow} & s(\mathbb{N}_a; X) \\
\uparrow \tau_1 & & \uparrow \tau_1 \\
s(\mathbb{N}_{a+1}; X) & \xrightarrow[{}_{GZ}\nabla_a^{\alpha}]{} & s(\mathbb{N}_{a+1}; X)
\end{array}
$$

In other words, we define the fractional difference operator $\Delta^\alpha : s(\mathbb{N}_0; X) \to s(\mathbb{N}_0; X)$ of order $\alpha > 0$ (in the sense of Riemann-Liouville) by

$$\Delta^\alpha f(n) := \Delta_0^m \circ \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m, \quad m := \lceil \alpha \rceil$.

We summarize the above diagram in the following result.

Theorem

Let $\alpha > 0$ and $a \in \mathbb{R}$ be given. The following assertions hold:

(i) $\tau_{a+m-\alpha} \circ \Delta_a^\alpha = \Delta^\alpha \circ \tau_a$, *where* $m := \lceil \alpha \rceil$.

(ii) $\tau_a \circ \nabla_a^\alpha = \Delta^\alpha \circ \tau_a$.

(iii) $\tau_{a+1} \circ_{GZ} \nabla_a^\alpha = \Delta^\alpha \circ \tau_{a+1}$.

We can introduce the notion of fractional difference in the sense of Caputo for the different notions of fractional sums.

Definition

Let $\alpha > 0$. The *α-th fractional Caputo like difference* is defined by

(a)
$$_C\Delta_a^\alpha f(t) = \Delta_a^{-(m-\alpha)}(\Delta_a^m f)(t), \quad t \in \mathbb{N}_{a+(m-\alpha)};$$

(b)
$$_{CGZ}\nabla_a^\alpha f(t) = _{GZ}\nabla_a^{-(m-\alpha)}(\Delta_{a+1}^m f)(t), \quad t \in \mathbb{N}_{a+1};$$

(c)
$$_C\nabla_a^\alpha f(t) = \nabla_a^{-(m-\alpha)}(\Delta_a^m f)(t), \quad t \in \mathbb{N}_a;$$

where $m-1 < \alpha < m$, $m = \lceil \alpha \rceil$.

We remark that the part (a) was proposed by Annastasiou. The others definitions seem to be new. We note that all are related by simple translation with (c) in case $a = 0$, that we denote by ${}_C\Delta^{\alpha}$. In other words, we define the fractional difference (in the sense of Caputo) of order $\alpha > 0$ by

$$ {}_C\Delta^{\alpha} f(n) := \nabla_0^{-(m-\alpha)}(\Delta_0^m f)(n), \quad n \in \mathbb{N}_0, \qquad (0.3) $$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

Example

For $0 < \alpha < 1$ we have $m = 1$ and

$$_C\Delta^{\alpha} f(n) := \Delta \circ \Delta^{-(1-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

and for $f(n) = n + 1$ we have

$$\begin{aligned} _C\Delta^{1} f(n) &= 1 \\ _C\Delta^{1/4} f(n) &= \frac{\Gamma(n + 11/4)}{\Gamma(7/4)\Gamma(n+2)} \\ _C\Delta^{1/2} f(n) &= \frac{\Gamma(n + 5/2)}{\Gamma(3/2)\Gamma(n+2)} \\ _C\Delta^{3/4} f(n) &= \frac{\Gamma(n + 9/4)}{\Gamma(5/4)\Gamma(n+2)} \end{aligned}$$

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Theorem

The following properties hold.

(a) *For* $0 < \alpha < \beta$*, we have*

$$\Delta^\alpha k^\beta(n) = k^{\beta-\alpha}(n+m), \quad n \in \mathbb{N}_0.$$

(b) *For any* $\alpha > 0$

$$\Delta^\alpha k^\alpha(n) = 0, \quad n \in \mathbb{N}_0.$$

(c) *For any* $\alpha > 0$ *the following holds*

$$\Delta^{-\alpha}(\Delta f)(n) = \Delta(\Delta^{-\alpha} f)(n) - k^\alpha(n+1)f(0), \quad n \in \mathbb{N}_0.$$

Example

The $1/2$-fractional difference of a constant sequence $1 \equiv k^1(j)$ is not zero. Indeed,

$$\Delta^{1/2} k^1(n) = k^{1/2}(n+1) = \frac{(2n+2)!}{4^{n+1}((n+1)!)^2}, \quad n \in \mathbb{N}_0.$$

Part (c) gives an interesting relation between the fractional difference in the sense of Caputo and Riemann-Liouville.

Corollary

For each $0 < \alpha < 1$ *and* $u \in s(\mathbb{N}_0; X)$, *we have*

$$_{C}\Delta^{\alpha} u(n) = \Delta^{\alpha} u(n) - k^{1-\alpha}(n+1)u(0), \quad n \in \mathbb{N}_0.$$

We recall that the *Z-transform* of a vector-valued sequence $f \in s(\mathbb{N}_0; X)$, which is identically zero for negative integers n, is defined by

$$\tilde{f}(z) \equiv Z[f(n)] = \sum_{j=0}^{\infty} z^{-j} f(j) \tag{0.4}$$

where z is a complex number.

Theorem

The following properties hold:

(a) *For all* $\alpha > 0$,

$$\widetilde{k^{\alpha}}(z) = \frac{z^{\alpha}}{(z-1)^{\alpha}} \quad \textit{for} \quad |z| > 1.$$

(b) *For all* $\alpha > 0$,

$$\widetilde{\Delta^{-\alpha}} f(z) = \frac{z^{\alpha}}{(z-1)^{\alpha}} \tilde{f}(z).$$

(c) *For* $0 < \alpha \leq 1$,

$$\widetilde{\Delta^{\alpha}} f(z) = z^{1-\alpha}(z-1)^{\alpha} \tilde{f}(z) - zf(0).$$

In the sequel, we introduce the following definition.

Definition

Let $\alpha, \beta > 0$ and $\lambda \in \mathbb{C}$ be given. The discrete Mittag-Leffler sequence is defined by

$$\mathcal{E}_{\alpha,\beta}(\lambda, n) = \frac{1}{2\pi i} \int_C \frac{z^{n+\beta-1}(z-1)^{\alpha-\beta}}{(z-1)^{\alpha} - \lambda z^{\alpha-\lceil \alpha \rceil}} dz, \quad n \in \mathbb{N}_0,$$

where C is a circle that encloses all poles of $(z-1)^{\alpha} - \lambda z^{\alpha-\lceil \alpha \rceil}$.

For example, for $\alpha = \beta = 1$ we obtain by Cauchy's formula $\mathcal{E}_{1,1}(\lambda, n) = (1+\lambda)^n$. It is well known that it corresponds to the discrete counterpart of the exponential function. A second example is given by $\alpha = 2$ and $\beta = 1$, obtaining

$$\mathcal{E}_{2,1}(\lambda, n) = \frac{1}{2}[(1+\sqrt{\lambda})^n + (1-\sqrt{\lambda})^n], \quad n \in \mathbb{N}_0,$$

which corresponds to the discrete counterpart of the cosine function

Theorem

The unique solution of the fractional difference equation of order $\alpha \in (0,1)$

$$ {}_C\Delta^{\alpha} u(n) = \lambda u(n), \quad n \in \mathbb{N}_0 $$

with initial condition $u(0) = u_0$ *is given by*

$$ u(n) = \mathcal{E}_{\alpha,1}(\lambda, n) u_0, \quad n \in \mathbb{N}_0. $$

For a given $c \in \mathbb{C}$, we denote $\mathbf{c} : \mathbb{N}_0 \to X$ by $\mathbf{c}(n) = c$ for all $n \in \mathbb{N}_0$.

Proposition

The problem

$$\Delta^\alpha u(n) = \mathbf{c}, \quad 0 < \alpha \leq 1,$$

with initial condition $u(0)$ *has the unique solution*

$$u(n) = k^\alpha(n)[u(0) + \frac{an}{\alpha}], \quad n \in \mathbb{N}.$$

In particular, for $\alpha = 1$, we recover a well known result

$$u(n) = u(0) + cn,$$

and for $\alpha = 1/2$, we obtain

$$u(n) = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\, n!}[u(0) + 2cn]. \tag{0.5}$$

Corollary

Let $T \in \mathbb{N}$ be given. The problem

$$\Delta^{\alpha} u(n) = \mathbf{c}, \quad 0 < \alpha < 1,$$

with boundary condition $u(0) = u(T)$ has the unique solution

$$u(n) = \frac{ak^{\alpha}(n)}{\alpha}\Big[\frac{T\,k^{\alpha}(T)}{1 - k^{\alpha}(T)} + \frac{an}{\alpha}\Big], \quad n \in \mathbb{N}.$$

Theorem

Let $0 < \alpha \leq 1$, $\lambda \in \mathbb{C}$ *and* $f \in s(\mathbb{N}_0; X)$ *be given. The solution of the problem*

$$\Delta^\alpha u(n) = \lambda u(n) + f(n+1)$$

is given by

$$u(n) = \mathcal{E}_{\alpha,\alpha}(\lambda, n)[u(0) - f(0)] + \sum_{k=0}^{n} \mathcal{E}_{\alpha,\alpha}(\lambda, n-k) f(k), \quad n \in \mathbb{N}_0.$$

Example

We consider the Gompertz fractional difference equation:

$$\Delta^{\alpha} u(n) = (b-1)u(n) + a, \quad n \in \mathbb{N}_0.$$

In the case $\alpha = 1$, the Gompertz difference equation describes the growth of tumors, where the parameter a is the growth rate and b is the exponential rate of growth deceleration. The fractional model was proposed in 2010. By the above Theorem, we obtain the solution

$$u(n) = \mathcal{E}_{\alpha,\alpha}(b-1, n)[u(0) - a] + a \sum_{k=0}^{n} \mathcal{E}_{\alpha,\alpha}(b-1, k), \quad n \in \mathbb{N}_0,$$

for the range $0 < \alpha \leq 1$.

In particular, in the case $\alpha = 1$ and since $\mathcal{E}_{1,1}(b-1, n) = b^n$, we obtain

$$u(n) = u(0)b^n + \frac{ab^n}{b-1} - \frac{a}{b-1}, \quad n \in \mathbb{N}_0.$$

Finally, we consider semilinear problems of the form

$$\Delta^{\alpha} u(n) = f(n, u(n)), \quad n \in \mathbb{N}_0,$$

where $0 < \alpha \leq 1$ and $f : \mathbb{N}_0 \times X \to X$ is given.

Theorem

Let $0 < \alpha \leq 1$. A sequence $u \in s(\mathbb{N}_0; X)$ is a solution of

$$\Delta^{\alpha} u(n) = f(n, u(n)), \quad n \in \mathbb{N}_0,$$

if and only if, u satisfies

$$u(n+1) = \sum_{k=0}^{n} k^{\alpha}(n-k) f(k, u(k)) + k^{\alpha}(n+1) u(0), \quad n \in \mathbb{N}_0.$$

Thank you!

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