Ergodic theory and linear dynamics

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Ergodic theory

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measurable map.

Definition

(a) T is called a **measure-preserving transformation** if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$.

(b) T is called **ergodic** if it is measure-preserving and satisfies one of the following equivalent conditions:

  (i) Given any measurable sets A, B with positive measures, one can find an integer $n \geq 0$ such that $T^n(A) \cap B \neq \varnothing$;

  (ii) if $A \in \mathcal{B}$ satisfies $T(A) \subset A$, then $\mu(A) = 0$ or 1.

Example - Irrational rotations

Let $X = \mathbb{T}$, let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{T}$ and let μ be the Lebesgue measure on $\mathbb{T}$. Let $\theta \in \mathbb{R}\backslash\mathbb{Q}$. Then

$$\begin{array}{rcl} T : (X, \mathcal{B}, \mu) & \to & (X, \mathcal{B}, \mu) \\ z & \mapsto & e^{i2\pi\theta} z \end{array}$$

is ergodic.

Example - Dyadic transformation

Let $X = [0,1)$, let $\mathcal{B}$ be the Borel σ-algebra on X and let μ be the Lebesgue measure on X. Then

$$\begin{aligned} T : (X, \mathcal{B}, \mu) &\rightarrow (X, \mathcal{B}, \mu) \\ x &\mapsto 2x \mod 1 \end{aligned}$$

is ergodic.

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μ_k is defined by $\mu_k(\{1\}) = 1/2$ and $\mu_k(\{-1\}) = 1/2$. Let $\mu = \cdots \otimes \mu_{-1} \otimes \mu_0 \otimes \mu_1 \otimes \dots$, namely

$$\mu\big(\{x \in X;\ (x_n,\dots,x_p) \in E_n \times \cdots \times E_p\}\big) = \mu_n(E_n) \times \cdots \times \mu_p(E_p).$$

Then

$$\begin{array}{rcl} T : (X,\mathcal{B},\mu) & \to & (X,\mathcal{B},\mu) \\ (x_n) & \mapsto & (x_{n+1}) \end{array}$$

is ergodic.

Why?

STRATEGY: Let $T \in \mathfrak{L}(X)$. Suppose that we can construct a measure μ defined on the Borel σ-algebra $\mathcal{B}$ of X, such that

- μ has **full support** (namely $\mu(U) > 0$ for any non-empty open set $U \subset X$);
- T is ergodic with respect to μ (namely $\mu(A)\mu(B) \neq 0$ implies that there exists $n \in \mathbb{N}$ with $T^n(A) \cap B \neq \varnothing$.)

Then T is topologically transitive, hence hypercyclic.

Birkhoff's ergodic theorem

Theorem

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure-preserving ergodic transformation. For any $f \in L^1(X, \mu)$,

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \xrightarrow{N\to\infty} \int_X f d\mu \quad \mu\text{-a.e.}$$

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Corollary

Let $T \in \mathfrak{L}(X)$. Assume that T is an ergodic transformation with respect to a Borel probability measure μ on X with full support. Then μ-almost every point $x \in X$ has the following property: for every non-empty open set $V \subset X$, one has

$$\liminf_{N\to\infty} \frac{\mathrm{card}\,\{n \in [0, N);\ \ T^n(x) \in V\}}{N} > 0\,.$$

Thus, if we can construct a probability measure μ defined on the Borel σ-algebra $\mathcal{B}$ of X, such that

- μ has **full support** (namely $\mu(U) > 0$ for any non-empty open set $U \subset X$);
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Hypercyclic operators satisfying the stronger condition (1) are called **frequently hypercyclic**. Moreover, we get that the set of (frequently) hypercyclic vectors is large in a probabilistic sense.

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- Given $T \in \mathfrak{L}(X)$ and a measure μ on X, how to prove that T is ergodic with respect to μ?
- What kind of conditions on $T \in \mathfrak{L}(X)$ ensures that we can construct a measure μ on X such that the dynamical system (T, μ) is ergodic?

Real Gaussian variables

For any $\sigma > 0$, let us denote by γ_σ the centered Gaussian measure on $\mathbb{R}$ with variance σ^2; that is,

$$d\gamma_\sigma = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{t^2}{2\sigma^2}}\, dt\,.$$

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$$\left.\begin{array}{l} X_1 \sim \gamma_{\sigma_1} \\ X_2 \sim \gamma_{\sigma_2} \\ X_1 \text{ and } X_2 \text{ are independent} \end{array}\right\} \implies X_1+X_2 \sim \gamma_\sigma \text{ with } \sigma^2 = \sigma_1^2+\sigma_2^2.$$

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If (X_n) are independent random variables, $X_n \sim \gamma_{\sigma_n}$ and if $\sum_n \sigma_n^2 < +\infty$, then $\sum_n X_n \sim \gamma_\sigma$ with $\sigma^2 = \sum_n \sigma_n^2$.

Complex Gaussian variables

Definition

A complex-valued random variable $Z : \Omega \to \mathbb{C}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to have **complex symmetric Gaussian distribution** if either Z is almost surely 0, or the real and imaginary parts of Z are independent and have centered Gaussian distribution with the same variance.

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Remark

Let Z be a complex random variable with complex Gaussian distribution, $\mathbb{E}(|Z|^2) = \sigma^2$, and write it $Z = X + iY$. Then $\mathbb{E}(|X|^2) = \sigma^2/2$ and $\mathbb{E}(|Y|^2) = \sigma^2/2$. If $\mathbb{E}|Z|^2 = 1$, then Z is said to be **standard**.

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Remark

Suppose that Z has complex symmetric Gaussian distribution. Then, for any $\lambda \in \mathbb{C}$, λZ has complex symmetric Gaussian distribution and $\mathbb{E}(|\lambda Z|^2) = |\lambda|^2\mathbb{E}(|Z|^2)$.

Gaussian measures

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(g_n)_{n\in\mathbb{N}}$ be a sequence of independent standard complex Gaussian variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Such a sequence (g_n) will be called a **standard Gaussian sequence**.

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Definition

A **Gaussian sum** in X is a sum $\sum_n g_n x_n$ where (x_n) is a sequence of vectors in X.

Convergence of Gaussian sums

Proposition

Let $(x_n) \subset X$. The following are equivalent :

(i) The Gaussian sum $\sum_n g_n x_n$ converges almost surely.

(ii) The Gaussian sum $\sum_n g_n x_n$ converges in L^p, for some $p \in [1, +\infty)$;

(iii) The Gaussian sum $\sum_n g_n x_n$ converges in L^p, for any $p \in [1, +\infty)$;

(iv) The Gaussian sum $\sum_n g_n x_n$ converges in probability.

If X is a Hilbert space, this is equivalent to $\sum_n \|x_n\|^2 < +\infty$.

Distribution of a Gaussian sum

The distribution of the Gaussian sum $\sum_n g_n x_n$ is defined by

$$\mu(A) = \mathbb{P}\left(\left\{\omega \in \Omega;\ \sum_n g_n(\omega) x_n \in A\right\}\right).$$

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- Step 1: the same result on $\mathbb{R}$.
  If (X_n) are independent random variables, $X_n \sim \gamma_{\sigma_n}$ and if $\sum_n \sigma_n^2 < +\infty$, then $\sum_n X_n \sim \gamma_\sigma$ with $\sigma^2 = \sum_n \sigma_n^2$.

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- Step 2: the same result on $\mathbb{C}$.

Lemma

Let (a_n) be a sequence of complex numbers such that $\sum_n |a_n|^2 < +\infty$. Then $\sum_n g_n a_n$ has complex symmetric Gaussian distribution.

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Theorem

If the series $\sum_n g_n x_n$ is almost surely convergent, then its distribution is a Gaussian measure.

Proof.

Let $x^* : (X, \mathcal{B}, \mu) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ and let $A \in \mathcal{B}(\mathbb{C})$. Does x^* has symmetric complex Gaussian distribution?

$$\begin{aligned} \mu(x^{*-1}(A)) &= \mathbb{P}\left(\left\{\omega \in \Omega;\ \left\langle x^*, \sum_n g_n x_n \right\rangle \in A\right\}\right) \\ &= \mathbb{P}\left(\left\{\omega \in \Omega;\ \sum_n g_n \langle x^*, x_n\rangle \in A\right\}\right) \end{aligned}$$

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...and by the lemma $\sum_n g_n \langle x^*, x_n\rangle$ has complex symmetric Gaussian distribution. □

The covariance operator

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Theorem

Let μ be a Gaussian measure on X.

(1) *One can define a continuous, conjugate-linear operator $R_\mu : X^* \to X$ such that, for every $(x^*, y^*) \in X^* \times X^*$:*

$$\langle R_\mu(x^*), y^* \rangle = \int_X \overline{\langle x^*, z \rangle} \langle y^*, z \rangle \, d\mu(z) = \langle x^*, y^* \rangle_{L^2(\mu)} .$$

The operator R_μ is called the **covariance operator** *of μ.*

(2) *Two Gaussian measures on X coincide if and only if they have the same covariance operator.*

Factorisation of the covariance operator

$$\langle R_\mu(x^*), y^* \rangle = \int_X \overline{\langle x^*, z \rangle} \, \langle y^*, z \rangle \, d\mu(z) = \langle x^*, y^* \rangle_{L^2(\mu)} \, .$$

- R_μ is symmetric: $\langle R_\mu(x^*), y^* \rangle = \overline{\langle x^*, R_\mu(y^*) \rangle}$;
- R_μ is positive: $\langle R_\mu(x^*), x^* \rangle \geq 0$.

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- R_μ is symmetric: $\langle R_\mu(x^*), y^*\rangle = \overline{\langle x^*, R_\mu(y^*)\rangle}$;
- R_μ is positive: $\langle R_\mu(x^*), x^*\rangle \geq 0$.

There exists some separable Hilbert space $\mathcal{H}$ and an operator $K : \mathcal{H} \to X$ such that

$$R_\mu = KK^*.$$

Covariance operator and properties of μ

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Theorem

Let μ be a Gaussian measure and let $R = KK^*$ be its covariance operator. Then the following properties are equivalent :

(i) *μ has full support;*

(ii) *R is one-to-one;*

(iii) *K has dense range.*

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The difficult implication: $(iii) \implies (i)$.

Gaussian measures with full support

K has dense range $\implies$ μ has full support.

Step 1 The support of a Gaussian sum.

Let $\sum_n g_n x_n$ be an almost surely convergent Gaussian sum and let ν be its distribution. Let $O \subset X$ be open such that $O \cap \mathrm{span}(x_n) \neq \varnothing$. Then $\nu(O) > 0$.

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The support of the distribution of a Gaussian sum $\sum_n g_n x_n$ is the closure of $\mathrm{span}(x_n)$.

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Step 2 The covariance operator of a Gaussian sum.

Let $\sum_n g_n x_n$ be an almost surely convergent Gaussian sum and let ν be its distribution. Then the covariance operator of ν is given by

$$R_\nu(x^*) = \sum_n \overline{\langle x^*, x_n \rangle}\, x_n.$$

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$$\begin{aligned}
\langle R_\nu(x^*), y^*\rangle &= \int_X \overline{\langle x^*, x\rangle}\langle y^*, x\rangle d\nu(x) \\
&= \int_\Omega \overline{\left\langle x^*, \sum_n g_n(\omega) x_n \right\rangle} \left\langle y^*, \sum_n g_n(\omega) x_n \right\rangle d\mathbb{P}(\omega) \\
&= \sum_{n,m} \overline{\langle x^*, x_n\rangle}\langle y^*, x_m\rangle \int_\Omega \overline{g_n(\omega)} g_m(\omega) d\mathbb{P}(\omega) \\
&= \sum_n \overline{\langle x^*, x_n\rangle}\langle y^*, x_n\rangle
\end{aligned}$$

Gaussian measures with full support

Step 3 Any Gaussian measure is the distribution of a Gaussian sum

Let $R = KK^*$ be the covariance operator of some Gaussian measure μ. Then, for any orthonormal basis (e_n) of $\mathcal{H}$,

$$Rx^* = \sum_n \overline{\langle x^*, Ke_n \rangle}\, Ke_n.$$

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$$\begin{aligned} R(x^*) &= K\left(\sum_{n=0}^{\infty}\langle e_n, K^*(x^*)\rangle_{\mathcal{H}}\, e_n\right) \\ &= \sum_{n=0}^{\infty}\overline{\langle x^*, K(e_n)\rangle} K(e_n)\,, \end{aligned}$$

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K has dense range $\Longrightarrow$ μ has full support.

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Measure-preserving transformations

Proposition

Let μ be a Gaussian measure on X with covariance operator R and let $T \in \mathfrak{L}(X)$. Then the image measure $\mu_T = \mu \circ T^{-1}$ is a Gaussian measure on X, with covariance operator TRT^*. In particular, T is measure-preserving if and only if $TRT^* = R$.

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$$\begin{aligned}
\langle R_T(x^*), y^* \rangle &= \int_Y \overline{\langle x^*, z \rangle} \langle y^*, z \rangle \, d\mu_T(z) \\
&= \int_X \overline{\langle x^*, T(z) \rangle} \langle y^*, T(z) \rangle \, d\mu(z) \\
&= \langle RT^* x^*, T^* y^* \rangle \\
&= \langle TRT^*(x^*), y^* \rangle .
\end{aligned}$$

What about ergodicity?

$T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic provided

$$\forall (A, B) \in \mathcal{B},\ \mu(A)\mu(B) \neq 0 \implies \exists n \in \mathbb{N},\ T^n(A) \cap B \neq \varnothing.$$

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Birkhoff's theorem says that for an ergodic transformation T,

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \xrightarrow{N \to \infty} \int_X f d\mu \quad \mu\text{-a.e.}$$

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Thus, $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic

$$\iff \forall f, g \in L^2(X, \mu), \frac{1}{N} \sum_{n=0}^{N-1} \int_X f(T^n x) g(x) d\mu \xrightarrow{N \to \infty} \int_X f d\mu \int_X g d\mu,$$

What about ergodicity?

$T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic provided

$$\forall (A, B) \in \mathcal{B},\ \mu(A)\mu(B) \neq 0 \implies \exists n \in \mathbb{N},\ T^n(A) \cap B \neq \varnothing.$$

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$$\iff \forall A, B \in \mathcal{B}, \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mu(T^n A \cap B) = \mu(A)\mu(B).$$

Weakly mixing - Strongly mixing

$T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is **weakly mixing**

$$\iff \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mu(A \cap T^{-n}(B)) - \mu(A)\mu(B)| = 0 \ (A, B \in \mathcal{B})$$

$$\iff \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \int_X f(T^n z) g(z)\, d\mu(z) - \int_X f d\mu \int_X g d\mu \right| = 0.$$

$T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is **strongly mixing**

$$\iff \lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B) \quad (A, B \in \mathcal{B})$$

$$\iff \lim_{n\to\infty} \int_X f(T^n z) g(z)\, d\mu(z) = \int_X f\, d\mu \int_X g\, d\mu \ (f, g \in L^2(X, \mu)).$$

The characterization of mixing properties

Theorem (Rudnicki, 1993)

Let μ be a Gaussian measure on X with full support and covariance operator R. Let $T \in \mathfrak{L}(X)$ be measure-preserving. The following are equivalent:

(i) *T is weakly mixing (strongly mixing, respectively) with respect to μ;*

(ii) *For all $x^*, y^* \in X^*$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle RT^{*n}(x^*), y^* \rangle| = 0$$

$(\lim_{n\to\infty} \langle RT^{*n}(x^*), y^* \rangle = 0,$ *respectively*$)$.

The easy implication

(i) T is strongly mixing with respect to μ, namely

$$\lim_{n\to\infty}\int_X f(T^n z)g(z)\,d\mu(z)=\int_X f\,d\mu\int_X g\,d\mu\;\big(f,g\in L^2(X,\mu)\big).$$

(ii) For all $x^*,y^*\in X^*$, $\lim_{n\to\infty}\langle RT^{*n}(x^*),y^*\rangle=0$.

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(ii) For all $x^*, y^* \in X^*$, $\lim_{n\to\infty} \langle RT^{*n}(x^*), y^* \rangle = 0$.

$$\langle R(x^*), y^* \rangle = \int_X \overline{\langle x^*, z \rangle} \, \langle y^*, z \rangle \, d\mu(z).$$

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$$\langle RT^{*n}(x^*),y^*\rangle\to\int_X\overline{\langle x^*,z\rangle d\mu(z)}\int_X\overline{\langle y^*,z\rangle d\mu(z)}=0.$$

The difficult implication - I

Assumption:

$$\lim_{n\to\infty} \langle RT^{*n}(x^*), y^* \rangle = 0 \ (x^*, y^*) \in X^*.$$

Conclusion:

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B) \ (A, B \in \mathcal{B}). \tag{2}$$

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Definition

$A \subset X$ is a **cylinder set** if there exist $N \geq 1$, $(x_1^*, \dots, x_N^*)$ a family of independent vectors of X^* and $E \subset \mathbb{C}^N$ such that

$$A = \left\{ x \in X;\ \left(\langle x_1^*, x \rangle, \dots, \langle x_N^*, x \rangle \right) \in E \right\}.$$

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It suffices to testify (2) for A, B cylinder sets.

The difficult implication - II

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Cylinder sets are "finite-dimensional sets"

The difficult implication - II

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Cylinder sets are "finite-dimensional sets" $\implies$ we need a finite-dimensional lemma.

Lemma

Let (ν_n) be a sequence of Gaussian measures on some finite-dimensional Banach space $E = \mathbb{C}^N$, and let ν be a Gaussian measure on E with full support. Assume that $R_{\nu_n} \to R_\nu$ as $n \to \infty$. Then $\nu_n(Q) \to \nu(Q)$ for every Borel set $Q \subset E$.

The difficult implication - III

Assumption:

$$\lim_{n\to\infty} \langle RT^{*n}(x^*), y^*\rangle = 0 \ (x^*, y^*) \in X^*.$$

$$\begin{aligned} A &= \left\{x \in X;\ \left(\langle x_1^*, x\rangle, \dots, \langle x_N^*, x\rangle\right) \in E\right\} \\ B &= \left\{x \in X;\ \left(\langle y_1^*, x\rangle, \dots, \langle y_M^*, x\rangle\right) \in F\right\}. \end{aligned}$$

Aim :

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B).$$

Tool :

Lemma

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- $T \in \mathfrak{L}(X)$ is strongly-mixing if and only if

$$\lim_{n\to\infty} \langle RT^{*n}(x^*), y^* \rangle = 0 \ (x^*, y^*) \in X^*.$$

Questions to solve

- What kind of measures shall we consider?
- Given $T \in \mathfrak{L}(X)$ and a measure μ on X, how to prove that T is a measure-preserving transformation?
- Given $T \in \mathfrak{L}(X)$ and a measure μ on X, how to prove that T is ergodic with respect to μ?
- What kind of conditions on $T \in \mathfrak{L}(X)$ ensures that we can construct a measure μ on X such that the dynamical system (T, μ) is ergodic?

Having sufficiently many $\mathbb{T}$-eigenvectors

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A vector $x \in X$ is a $\mathbb{T}$-**eigenvector** for T if $T(x) = \lambda x$ for some $\lambda \in \mathbb{T}$.

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A map $E : \mathbb{T} \to X$ is a **perfectly spanning $\mathbb{T}$-eigenvector field** provided

(i) $E \in L^\infty(\mathbb{T}, X)$;

(ii) $\forall \lambda \in \mathbb{T},\ \ TE(\lambda) = \lambda E(\lambda)$;

(iii) For any $A \subset \mathbb{T}$ with $m(A) = 0$, then $\mathrm{span}\big(E(\lambda);\ \lambda \in A\big)$ is dense in A.

The operator K_E

Let $T \in \mathfrak{L}(X)$ with a perfectly spanning $\mathbb{T}$-eigenvector field E. Define

$$\begin{aligned} K_E : L^2(\mathbb{T}, dm) &\rightarrow X \\ f &\mapsto \int_{\mathbb{T}} f(\lambda)E(\lambda)dm(\lambda) \end{aligned}$$

and $R = K_E K_E^*$. Then

1. K_E has dense range;
2. $TRT^* = T$;
3. For any $x^*, y^* \in X^*$, $\langle RT^{*n}(x^*), y^* \rangle \rightarrow 0$.

The intertwining equation

$$\begin{array}{rcl} K_E : L^2(\mathbb{T}, dm) & \to & X \\ f & \mapsto & \int_{\mathbb{T}} f(\lambda)E(\lambda)dm(\lambda) \end{array}$$

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$$TK = KV.$$

$$\begin{aligned} TK_E(f) &= \int_{\mathbb{T}} f(\lambda)TE(\lambda)dm(\lambda) \\ &= \int_{\mathbb{T}} f(\lambda)\lambda E(\lambda)dm(\lambda) \\ &= KVf. \end{aligned}$$

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   Ok for R!

Is R the covariance operator of some Gaussian measure μ?

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Definition

Let $\mathcal{H}$ be a separable Hilbert space. An operator $K \in \mathfrak{L}(\mathcal{H}, X)$ is said to be γ-**radonifying** if for some (equivalently, for any) orthonormal basis (e_n) of $\mathcal{H}$, the Gaussian series $\sum g_n(\omega)K(e_n)$ converges almost surely.

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Proposition

Let $K \in \mathfrak{L}(\mathcal{H}, X)$ be γ-radonifying. Then $R = KK^*$ is the covariance operator of some Gaussian measure μ on X.

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Proposition

Let $K \in \mathfrak{L}(\mathcal{H}, X)$ be γ-radonifying. Then $R = KK^*$ is the covariance operator of some Gaussian measure μ on X.

It suffices to take for μ the distribution of the Gaussian sum $\sum_n g_n K e_n$.

On a Hilbert space

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Fact. On a Hilbert space, a Gaussian sum $\sum_n g_n x_n$ converges almost surely if and only if $\sum_n \|x_n\|^2 < +\infty$.

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It is Hilbert-Schmidt!

$R = K_E K_E^*$ is the covariance operator of some Gaussian measure!

Theorem on a Hilbert space

Theorem (B. Grivaux (2006))

Let X be a separable Hilbert space and let $T \in \mathfrak{L}(X)$ be such that T has a perfectly spanning $\mathbb{T}$-eigenvectorfield. Then there exists a Gaussian measure μ on X with full support, with respect to which T is a strongly-mixing measure-preserving transformation.

What about Banach spaces?

$K \in \mathfrak{L}(\mathcal{H}, X)$ is γ-radonifying if for some orthonormal basis (e_n) of $\mathcal{H}$, the Gaussian series $\sum g_n(\omega) K(e_n)$ converges almost surely.

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Definition

A Banach space X is said to have (Gaussian) **type** $p \in [1, 2]$ if

$$\left\| \sum_n g_n x_n \right\|_{L^2(\Omega,X)} \leq C \left(\sum_n ||x_n||^p \right)^{\frac{1}{p}},$$

for some finite constant C and every finite sequence $(x_n) \subset X$.

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- A Hilbert space has type 2;
- L^p-spaces have type $\min(p, 2)$;
- Any Banach space has type 1;

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Corollary

Let X be a Banach space with type p and let $K \in \mathfrak{L}(\mathcal{H}, X)$. Then K is γ-radonifying as soon as $\sum_n \|Ke_n\|^p < +\infty$ for some orthonormal basis (e_n) of $\mathcal{H}$.

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Theorem (B. Matheron, 2009)

Let X be a separable Banach space and let $T \in \mathfrak{L}(X)$ be such that T has a perfectly spanning $\mathbb{T}$-eigenvector field E. Suppose moreover that :

- *X has type p;*
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- $(e_n) = (e^{int})_{n\in\mathbb{Z}}$ (B. Grivaux 2007). The result is less good.

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- $(e_n) = (e^{int})_{n\in\mathbb{Z}}$ (B. Grivaux 2007). The result is less good.
- $(e_n) =$ the Haar basis of $L^2(\mathbb{T})$.

Banach spaces=Hilbert spaces!

Theorem (B. 2011)

Let X be a separable Banach space and let $T \in \mathfrak{L}(X)$ be such that T has a perfectly spanning $\mathbb{T}$-eigenvector field. Then there exists a Gaussian measure μ on X with full support, with respect to which T is a weakly-mixing measure-preserving transformation.

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Strategy. Instead of considering

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with σ a continuous measure on $\mathbb{T}$. σ will be carried on Cantor set!

Cantor sets

Definition

A subset $\mathcal{C}$ of $\mathbb{T}$ is a **Cantor set** if it is the continuous image of $\{-1, 1\}^{\mathbb{N}}$.

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$\{-1,1\}^{\mathbb{N}}$ will be endowed with its Haar measure ν. ν is the tensor product $\mathbb{P}_1 \otimes \mathbb{P}_2 \otimes \dots$, with, one each coordinate,

$$\mathbb{P}_k(\{-1\}) = 1/2 \text{ and } \mathbb{P}_k(\{1\}) = 1/2.$$

An orthonormal basis of $L^2(\{-1,1\}^\omega)$.

Any $\omega \in \{-1,1\}^{\mathbb{N}}$ can be written

$$\omega = (\varepsilon_1(\omega), \varepsilon_2(\omega), \dots).$$

Definition

Let $A \subset \mathcal{P}_f(\mathbb{N})$. The **Walsh function** w_A is defined by

$$w_A(\omega) = \prod_{n \in A} \varepsilon_n(\omega).$$

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Theorem

$(w_A)_{A \in \mathcal{P}_f(\mathbb{N})}$ is an orthonormal basis of $L^2(\{-1,1\}^{\mathbb{N}}, \nu)$.

A new γ-radonifying operator

Lemma

Let $\phi : \{-1,1\}^{\mathbb{N}} \to \mathcal{C}$ be an homeomorphism and let σ be the image of the Haar measure ν on $\{-1,1\}^{\mathbb{N}}$ by ϕ. Let $u : \{-1,1\}^{\mathbb{N}} \to X$ be a continuous function such that, for any $n \geq 1$, for any $(s_1,\dots,s_{n-1}) \in \{-1,1\}^{n-1}$, any $s', s'' \in \{-1,1\}^{\mathbb{N}}$,

$$\|u(s_1,\dots,s_{n-1},1,s') - u(s_1,\dots,s_{n-1},-1,s'')\| \leq 3^{-n}.$$

Let also $E = u \circ \phi^{-1}$. Then there exists an orthonormal basis (e_n) of $L^2(\mathbb{T}, d\sigma)$ such that the operator $K_E : L^2(\mathbb{T}, d\sigma) \to X$ satisfies

$$\sum_n \|K_E(e_n)\| < +\infty.$$

What remains to be done

- Prove that, if T admits a perfectly spanning $\mathbb{T}$-eigenvector field, then one can construct $\phi : \mathcal{C} \to \mathbb{T}$, $u : \mathcal{C} \to X$ such that

$$\|u(s_1, \ldots, s_{n-1}, 1, s') - u(s_1, \ldots, s_{n-1}, -1, s'')\| \leq 3^{-n}$$

and $u(s)$ is a $\mathbb{T}$-eigenvector with eigenvalue $\phi(s)$.

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- Prove that everything remains true with $K_E : L^2(\mathbb{T}, d\sigma) \to X$ instead of $K_E : L^2(\mathbb{T}, dm) \to X$.

In fact, we will have to consider several such maps instead of one!

The construction of Cantor sets

Lemma

Let $T \in \mathcal{L}(X)$ with a perfectly spanning $\mathbb{T}$-eigenvector field. Let also (ε_n) be a sequence of positive real numbers. There exist a sequence $(\mathcal{C}_i)$ of subsets of $\mathbb{T}$, a sequence of homeomorphisms (ϕ_i) from $\{-1,1\}^{\mathbb{N}}$ onto $\mathcal{C}_i$ and a sequence of continuous functions (u_i), $u_i : \{-1,1\}^{\mathbb{N}} \to S_X$ such that, setting $E_i = u_i \circ \phi_i^{-1}$,

(a) *for any $i \geq 1$ and any $\lambda \in \mathcal{C}_i$, $TE_i(\lambda) = \lambda E_i(\lambda)$;*

(b) $span(E_i(\lambda);\ i \geq 1,\ \lambda \in \mathcal{C}_i)$ *is dense in X;*

(c) *for any $n \geq 1$, any $(s_1, \dots, s_{n-1}) \in \{-1,1\}^{n-1}$, any $s', s'' \in \{-1,1\}^{\mathbb{N}}$,*

$$\|u_i(s_1, \dots, s_{n-1}, 1, s') - u_i(s_1, \dots, s_{n-1}, -1, s'')\| \leq \varepsilon_n.$$

How to prove this?

Step 1 Since E has a perfectly spanning $\mathbb{T}$-eigenvector field, there exists a sequence (x_i) satisfying :

- each x_i belongs to S_X, is a $\mathbb{T}$-eigenvector and the corresponding eigenvalues (λ_i) are all different;
- each x_i is a limit of a subsequence $(x_{n_k})_{k\geq 1}$;
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Step 2 The construction...

Proof of the main result

We apply the previous lemma with $\varepsilon_n = 3^{-n}$. We get $\mathcal{C}_i$, u_i, E_i, σ_i and an orthonormal basis of $L^2(\mathbb{T}, d\sigma_i)$ such that

$$\sum_n \|K_{E_i}(e_{n,i})\| < +\infty.$$

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We set $\mathcal{H} = \oplus_{i\geq 1} L^2(\mathbb{T}, d\sigma_i)$ and let $K : \mathcal{H} \to X$ be defined by

$$K(\oplus_i f_i) = \sum_i \alpha_i K_{E_i}(f_i)$$

where (α_i) satisfies

(a) $\sum_i \alpha_i^2 \|E_i\|^2_{L^2(\mathbb{T},\sigma_i,X)} < +\infty$

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Everything works with $R = KK^*$.

The correct statement

In fact, we have obtained the following statement:

Theorem

Let $T \in \mathfrak{L}(X)$ be such that, for any $D \subset \mathbb{T}$ countable, $\ker(T - \lambda I; \lambda \in \mathbb{T} \backslash D)$ *is a dense subset of X. Then there exists a Gaussian measure μ on X with full support, with respect to which T is a weakly-mixing measure-preserving transformation.*

Example - backward weighted shifts

Let $B_{\mathbf{w}}$ be the **weighted backward shift** on $\ell^p(\mathbb{N})$ with weight sequence (w_n):

$$B_{\mathbf{w}}(x_0, x_1, \dots) = (w_1 x_1, w_2 x_2, w_3 x_3, \dots).$$

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Suppose that

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There exists a Gaussian measure μ on $\ell^p(\mathbb{N})$ with full support, with respect to which $B_{\mathbf{w}}$ is a measure-preserving and weakly mixing transformation.

Example - backward weighted shifts

The condition

$$\sum_{n\geq 1} \frac{1}{(w_1 \cdots w_n)^p} < \infty \,.$$

ensures that $B_{\mathbf{w}}$ admit $\mathbb{T}$-eigenvectors:

$$E(\lambda) := \sum_{n\geq 0} \frac{\lambda^n}{w_1 \cdots w_n} \, e_n \,.$$

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This eigenvector field is perfectly spanning.

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$$\langle y, E(\lambda)\rangle = 0 \text{ a.e.}.$$

Then

$$g(\lambda) = \sum_n \frac{y_n}{w_1 \dots w_n} \lambda^n = 0 \text{ a.e.}.$$

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$$\implies \hat{g}(n) = 0 \text{ for all } n \in \mathbb{N}.$$

Example - Adjoints of multipliers

$$H^2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C}; \|f\|_{H^2}^2 := \sup_{r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \right\}$$
$$= \left\{ f(z) = \sum_n a_n z^n;\ \sum_n |a_n|^2 < +\infty \right\}.$$

$$H^\infty(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{C};\ \|f\|_\infty < +\infty \}.$$

Definition

For $\phi \in H^\infty(\mathbb{D})$, the multiplier M_ϕ is defined by $M_\phi(f) = \phi f$, $f \in H^2(\mathbb{D})$.

Theorem

If ϕ is non-constant and $\phi(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$, then there exists a Gaussian measure with full support on $H^2(\mathbb{D})$ with respect to which M_ϕ^* is a measure-preserving and weakly mixing transformation.

Adjoints of multipliers

Let k_z be the **reproducing kernel** at $z \in \mathbb{D}$:

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k_z is an eigenvector for M_ϕ^*.

$$\langle f, M_\phi^*(k_z) \rangle_{H^2} = \langle \phi f, k_z \rangle_{H^2} = \phi(z) f(z) = \langle f, \overline{\phi(z)} k_z \rangle_{H^2} .$$

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$$E(e^{i\theta}) := \mathbf{1}_I(e^{i\theta}) k_{\phi^{-1}(e^{i\theta})} .$$

is a (conjugate) $\mathbb{T}$-eigenvector field.

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$$f \equiv 0.$$

And so on...

- Many other examples (composition operators,...);
- Many other results (about the converse, on semigroups of operators,...)

Muchas gracias!