# Entropy, inverse limits and attractors

(joint work with Jan P. Boroński)

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# Basic setting

1. $X$ always compact metric space
2. if additionally connected then we say that $X$ is a continuum
3. $\#X > 1$ - nondegenerate
4. $f : X \to X$, always continuous
5. $C(X)$ - set of all such maps $f$
6. $(X, f)$ - a dynamical system
7. $I = [0, 1]$
8. $G$ - a topological graph
9. inverse limit - $\mathbb{X} = \varprojlim\{f, X\} = \{(x_0, x_1, \ldots) : x_i \in X, f(x_{i+1}) = x_i\}$
10. shift homeo. - $\sigma_f(x_0, x_1, \ldots) = (f(x_0), x_0, x_1, \ldots)$

# Indecomposable arc-like continua

1. continuum $C$ is arc-like if for every $\varepsilon > 0$ there is an $\varepsilon$-map $\pi\colon C \to I$ (i.e. $\operatorname{diam} \pi^{-1}(x) < \varepsilon$ for every $x \in I$).
2. continuum $C$ is indecomposable if is not the union of two proper subcontinua.
3. hereditarily indecomposable if all nondegenerate subcontinua are indecomposable.
4. arc-like hereditarily indecomposable continuum is topologically unique - we call it the pseudoarc (Knaster; Moise; Bing).

# Inverse limits and attractors

**Theorem (Barge & Martin)**

Every continuum $\mathbb{X} = \varprojlim\{f, [0,1]\}$, can be embedded into a disk $D$ in such a way that

(i) $\mathbb{X}$ is an attractor of a homeomorphism $h\colon D \to D$,

(ii) $h|_{\mathbb{X}} = \sigma_f$; i.e. $h$ restricted to $\mathbb{X}$ agrees with the shift homeomorphism induced by $f$, and

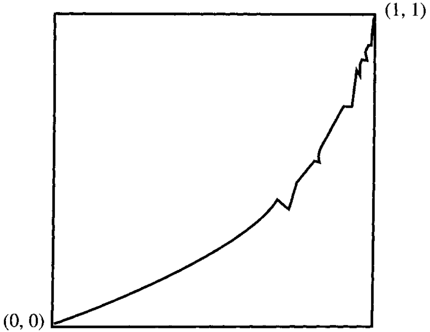
(iii) $h$ is the identity on the boundary of $D$.

**Remark**

It was pointed by Barge & Roe that the same is true if $f$ is a degree $\pm 1$ circle map and $h$ is an annulus homeomorphism.

# Pseudoarc

1. If $f \in C(I)$ has some special properties, then $\mathbb{X}$ is a pseudoarc.
2. Then we can study dynamical properties of the homeomorphism $\sigma_f$ in terms of $f$.
3. Example of G. Henderson (Duke Math. J., 1964):



# Method of Minc and Transue

1. We say that $f \in C(I)$ is $\delta$-crooked between $a$ and $b$ if,
   - for every two points $c, d \in I$ such that $f(c) = a$ and $f(d) = b$,
   - there is a point $c'$ between $c$ and $d$ and there is a point $d'$ between $c'$ and $d$
   - such that $|b - f(c')| < \delta$ and $|a - f(d')| < \delta$.
2. We say that $f$ is $\delta$-crooked if it is $\delta$-crooked between every pair of points.

**Theorem**

Let $f \in C(I)$ be a map with the property that,

- for every $\delta > 0$ there is an integer $n > 0$
- such that $f^n$ is $\delta$-crooked.

Then $\mathbb{X}$ is the pseudoarc.

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# Circle-like maps

1. We say that $\omega \colon I \to G$ is $\delta$-crooked if,
   - there are points $0 \leq c' < d' \leq 1$
   - such that $d(\omega(1), \omega(c')) < \delta$ and $d(\omega(0), \omega(d')) < \delta$.
2. We say that $f$ is $\delta$-crooked if every $\omega \colon I \to G$ is $\delta$-crooked.

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Let $f \in C(G)$ be a map with the property that,

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Then $\mathbb{X}$ is the hereditarily indecomposable.

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- Map $f \in C(X)$ is exact if for every open $U$ there is $n > 0$ such that $f^n(U) = X$.
- Using approximation technique of Minc-Transue it is possible to generate:
  1. *(Minc & Transue)* (topologically) mixing map of the pseudo-arc
  2. *(Kawamura, Tuncali & Tymchatyn)* mixing map of the pseudo-circle (or other continua from inverse limits)
- Every example of this kind, when transitive is automatically mixing (because of terminal periodic decomposition for transitive maps).
- $h_{\text{top}}(f) = h_{\text{top}}(\sigma_f)$ so all these examples have positive topological entropy
  3. *(Kościelniak, O. & Tuncali)* On pseudo-arc it is possible that $\sigma_f$ is mixing but not exact, on pseudo-circle it is always exact when mixing.
  4. *(O. & Drwięga)* Such example on pseudo-arc exists (i.e. mixing but not exact)

# Crookedness and dynamics

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# An old question

**Question (Barge, 1989?)**

Is every real number the entropy of some homeomorphism on the pseudo-arc?

**Theorem (Mouron, 2012)**

If $f \in C(I)$ is such that the inverse limit $\mathbb{X}$ is the pseudoarc then $h_{\text{top}}(f) \in \{0, \infty\}$.

- The answer to Barge's question is still unknown.
- With Example of Henderson + Minc and Transue technique we see that both cases 0, $\infty$ can be obtained in practice.

# Work in progress (with J. Boroński)

- We can prove the following (with other methodology than Mouron).

**Theorem**

If $f \in C(G)$ is such that the inverse limit $\mathbb{X}$ is the hereditarily indecomposable then $h_{\mathrm{top}}(f) \in \{0, \infty\}$.

- it is known that there is a homeomorphism of the pseudo-circle with zero entropy - example of M. Handel from 1982 - even a global attractor and minimal set for plane homeomorphism
- but can zero entropy shift homeomorphism $\sigma_f$ of the pseudo-circle be constructed?

**Theorem (still not all details sufficiently verified...)**

If $f \in C(G)$ is such that the inverse limit $\mathbb{X}$ is the hereditarily indecomposable and $h_{\mathrm{top}}(f) > 0$ then there exists a closed entropy set $A \subset [0, 1]$ such that $h_{\mathrm{top}}(A) = \infty$.

# Chaos in the sense of Li and Yorke

1. $(x, y)$ is Li-Yorke pair if is proximal but not asymptotic, i.e.
   - $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$,
   - $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$.
2. $S$ - scrambled, if every $x, y \in S$, $x \neq y$ is Li-Yorke pair.
3. $f$ - Li-Yorke chaotic if there exists uncountable scrambled set.
4. For $f \in C(I)$ we have Li-Yorke chaos:
   - when entropy of $f$ is positive, or equivalently there is a point of odd period,
   - when entropy is zero, for some (but not all) maps of type $2^\infty$, i.e. maps with points of period $2^n$ for every $n$.
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# Decomposable continua

1. A continuum is decomposable if it can be written as the union of two proper subcontinua.
2. It is hereditarily decomposable if every subcontinuum is decomposable.
3. It was recently proved that positive entropy implies Li-Yorke chaos, but
   - NO hereditarily decomposable arc-like continuum admits homeomorphisms with positive entropy (Mouron),
   - homeomorphisms of arc-like hereditarily decomposable continua admit only $2^n$-periodic orbits (Ye, Ingram).

**Question**

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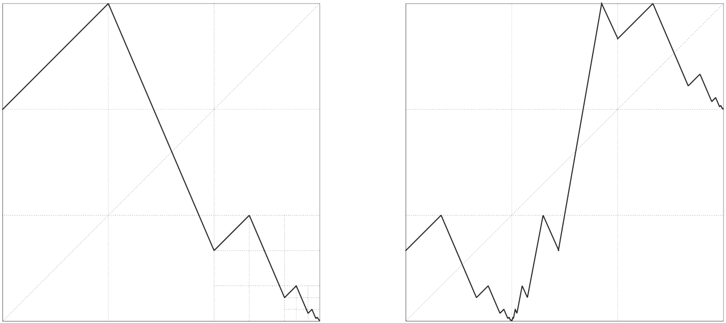
# Search for an example - first approximation

1. Graph of $f$ and $f^2$



2. Map $f$ is of type $2^\infty$. Its $\omega$-limit sets are either periodic points or an odometer (the unique infinite $\omega$-limit set).

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Inverse limit $\mathbb{X} = \varprojlim \{f, [0,1]\}$ is hereditarily decomposable continuum.

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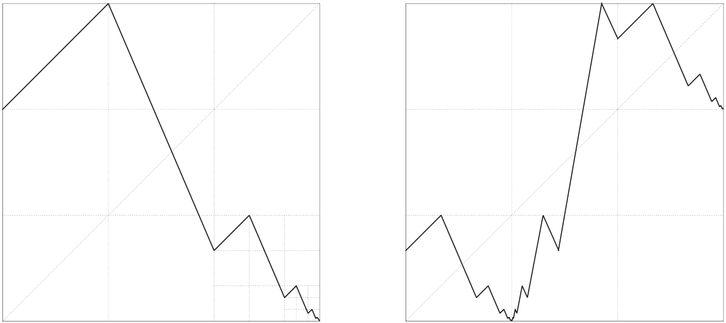

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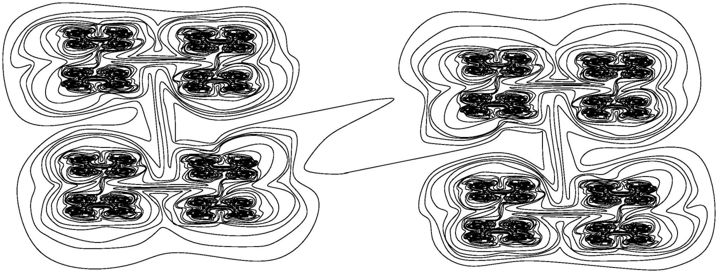
# Search for an example - final step

1. Blow up properly selected orbit of $f$ to introduce Li-Yorke pair (in new map $g$), but without introducing indecomposable subcontinuum into inverse limit (of $g$).



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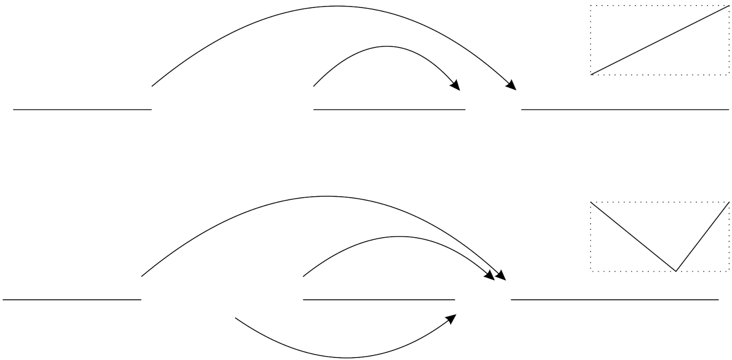
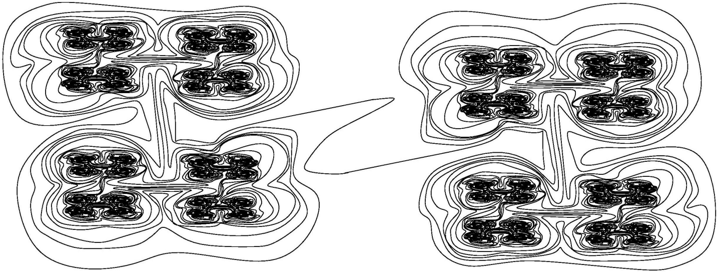


1. For our "Denjoy-type" construction we select a point in the infinite $\omega$-limit set (odometer) whose preimages do not contain turning point.

# Final remark

1. Our example is Suslinean (any family of pairwise disjoint and nondegenerate subcontinua is countable).
2. Embedding other "wandering" subcontinuum can make it non Suslinean.



# Related problems

There exists an arc-like hereditarily decomposable continuum that contains no arc (Nadler's book *Continuum theory. An introduction*, 1992)

**Question 1**

Is there a hereditarily decomposable arc-like continuum $X$ which contains no arc, admitting a Li-Yorke chaotic homeomorphism?

**Question 2**

Is there a Li-Yorke chaotic zero entropy homeomorphism of the pseudoarc?

The answer to Q2, if positive, cannot be obtained by inverse limit construction with one map. If a map $f \in C(I)$ has a periodic point of period 2 or larger, and $X_\varphi$ is the pseudoarc, then it has a periodic point of odd period other than one (Block, Keesling, Uspenskij, 2000).