Composition operators: the essential norm and norm-attaining

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The purpose of this talk is to first discuss the essential norm of composition operators acting between standard weighted Bergman spaces. In particular, I give an elementary proof for characterizing compactness of such operators. Moreover, I will discuss norm-attaining weighted composition operators on weighted Banach spaces of holomorphic functions, these spaces are sometimes called weighted Bergman spaces of infinite order.

We denote by $H(D)$ the space of holomorphic functions on the open unit disk D in the complex plane. For $1 \leq p < \infty$ and $\alpha > -1$, the ***standard weighted Bergman space*** $A^p_\alpha := A^p_\alpha(D)$ is the set of all analytic functions on D such that

$$||f||^p_{\alpha,p} = (\alpha+1)\int_D |f(z)|^p(1-|z|^2)^\alpha dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi}dxdy$ is the normalized Lebesgue area measure in D. For $1 \leq p < \infty$, this defines a norm, and hence A^p_α is a Banach space. From now on, let $dA_\alpha(z)$ denote the normalized area measure $(\alpha+1)(1-|z|^2)^\alpha \, dA(z)$.

The ***weighted Banach spaces of holomorphic functions***

$$H_v^\infty := \{f \in H(D);\ \|f\|_v = \sup_{z \in D} v(z)|f(z)| < \infty\}$$

and on the smaller spaces

$$H_v^0 := \{f \in H_v^\infty;\ \lim_{|z| \to 1} v(z)|f(z)| = 0\}$$

endowed with norm $\|.\|_v$, where the ***weight*** $v : D \to R$ is bounded, continuous and strictly positive function. These weighted Banach spaces of analytic functions have been studied intensively by several people during the last four decades. Especially I want to mention ***Bierstedt, Bonet, Lusky, Shields and Williams.***

Let $\varphi : D \to D$ be an analytic mapping and $\psi \in H(D)$. Each such pair (φ, ψ) induces via composition and multiplication a linear weighted composition operator

$$C_{\varphi,\psi}(f) = \psi(f \circ \varphi)$$

which preserves $H(D)$. If $\psi = 1$, then $C_{\varphi,1} = C_\varphi$ is the usual composition operator. Operators of this type have been studied on various spaces of analytic functions. For a discussion of composition operators on classical spaces of analytic functions we refer to the excellent monographs of ***Shapiro*** and ***Cowen-MacCluer***.

The reason why I am interested in studying the essential norm of composition operators between standard weighted Bergman spaces is that ”different” estimates of the essential norm of a composition operator have been obtained as you soon will see.

Recall that the ***essential norm*** $||T||_e$ of a bounded operator $T \in L(X, Y)$ is the distance in the operator norm from T to the compact operators, that is, $||T||_e = \inf\{||T - K|| : K$ is compact$\}$. Thus the essential norm of T equals zero if and only if T is compact.

It is known that the following quantities are comparable for $C_\varphi : A^p_\alpha \to A^p_\beta,\ p > 1, \alpha, \beta > -1$:

$$||C_\varphi||_e \simeq \limsup_{|z|\to 1} \left(\frac{N_{\varphi,2+\beta}(z)}{(1-|z|^2)^{2+\alpha}} \right)^{\frac{1}{p}}$$ (***Shapiro***, 1987)

$$\simeq \limsup_{|z|\to 1} \left(\int_D \frac{(1-|z|^2)^{\alpha+2}}{|1-\bar{z}\varphi(w)|^{2(\alpha+2)}}\, dA_\beta(w) \right)^{\frac{1}{p}}$$

(***Čučković and Zhao***, 2007),

where $N_{\varphi,2+\alpha}$ is the ***generalized Nevanlinna counting*** function defined by $N_{\varphi,2+\alpha}(x) = \sum_{z\in\varphi^{-1}(x)} (\log(1/|z|))^{2+\alpha},\ x \in D \setminus \{\varphi(0)\}$.

The result Čučković and Zhao is based on the Carleson measure theorem in Bergman spaces and Shapiro uses a technique involving the generalized Nevanlinna counting function to obtain his result. This means that both proofs are using advanced tools. These two estimates of the essential norm of a composition operator are in my opinion neat results, but in practice it is still pretty hard to evaluate the above mention integral as well to evaluate the expression involving the generalized Nevanlinna counting function.

So both methods used are somewhat complex!

Boundedness and the essential norm of composition operators

We now state our first main result.

Theorem 1. *Let* $p > 1$ *and* $\alpha,\ \beta > -1$. *Assume that* $\varphi : D \to D$ *is an analytic function such that*

$$\sup_{z \in D} \frac{(1-|z|^2)^{\beta+1}}{(1-|\varphi(z)|^2)^{\alpha+1}} < \infty.$$

Then the operator $C_\varphi : A^p_\alpha \to A^p_\beta$ *is bounded and*

$$\limsup_{|z|\to 1} \frac{(1-|z|^2)^{\frac{\beta+2}{p}}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \le ||C_\varphi||_e \lesssim \limsup_{|z|\to 1} \frac{(1-|z|^2)^{\frac{\beta+1}{p}}}{(1-|\varphi(z)|^2)^{\frac{\alpha+1}{p}}}.$$

If one uses the method on pages 140-141 in the book of Cowen-MacCluer, then the result of Shapiro involving the generalized Nevanlinna counting function gives our above mentioned result. Furthermore, when φ is univalent, it reduces to the formula

$$||C_\varphi||_e^p \approx \limsup_{|z|\to 1} \frac{(1-|z|^2)^{\beta+2}}{(1-|\varphi(z)|^2)^{\alpha+2}}.$$

On the contrary, our argument is straightforward and only based on Littlewood's subordination principle.

Indeed, as a direct consequence of Littlewood's subordination principle we get:

Lemma 1. *Let* $p > 0,\ 0 < r < 1$ *and* $f \in H(D)$. *Assume that* φ *is an analytic selfmap of* D *such that* $\varphi(0) = 0$. *Suppose that for some* $\rho \in (0,1)$ *holds* $|\varphi(re^{i\theta})| \le \rho$ *for every* $0 \le \theta < 2\pi$. *Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |(f\circ\varphi)(re^{i\theta})|^p \, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta.$$

Theorem 2. *Let* $p > 1$ *and* $\eta > 0$, $\lambda > -1$. *Assume that* $\varphi : D \to D$ *is analytic such that* $\varphi(0) = 0$ *and*

$$\sup_{z\in D} \frac{1-|z|^2}{(1-|\varphi(z)|^2)^\eta} < \infty.$$

Then the operator $C_\varphi : A^p_{(\lambda+1)\eta-1} \to A^p_\lambda$ *is bounded and*

$$||C_\varphi||_e^p \lesssim \left(\limsup_{|z|\to 1} \frac{1-|z|^2}{(1-|\varphi(z)|^2)^\eta} \right)^{\lambda+1}.$$

Proof. We only consider the statement concerning the essential norm.

Fix $0 < \delta < 1$ and set

$$M_\delta := \sup_{|z|>\delta} \frac{1-|z|^2}{(1-|\varphi(z)|^2)^\eta} > 0.$$

Define the function $\rho : [\delta, 1] \to [0, 1]$ by

$$\rho(r) = \left[1 - \frac{1}{M_\delta^{\frac{1}{\eta}}}(1-r^2)^{\frac{1}{\eta}}\right]^{\frac{1}{2}}.$$

By the definition of M_δ, $|\varphi(re^{i\theta})| \leq \rho(r)$ for every $0 \leq \theta < 2\pi$ and $\delta \leq r < 1$. Let $D_\delta = \{z \in D : |z| < \delta\}$ and $\gamma = (\lambda+1)\eta - 1$. It is easy to see that we have the following estimate

$$||C_\varphi||_e^p \lesssim \lim_{\delta \to 1} \sup_{||f||_{\gamma,p} \leq 1} \int_{D \setminus D_\delta} |f \circ \varphi(z)|^p dA_\lambda.$$

Let $f \in A^p_\gamma, ||f||_{\gamma,p} \leq 1$ be arbitrary. Applying our above lemma and Fubini's theorem, we get

$$\begin{aligned}\int_{D\setminus D_\delta} |f \circ \varphi(z)|^p dA_\lambda &= \frac{\lambda+1}{\pi} \int_\delta^1 \int_0^{2\pi} |(f \circ \varphi)(re^{i\theta})|^p \, d\theta r(1-r^2)^\lambda dr \\ &\leq \frac{\lambda+1}{\pi} \int_\delta^1 \int_0^{2\pi} |f(\rho(r)e^{i\theta})|^p \, d\theta r(1-r^2)^\lambda dr \\ &= \frac{\lambda+1}{\pi} \int_0^{2\pi} \int_\delta^1 |f(\rho(r)e^{i\theta})|^p r(1-r^2)^\lambda dr d\theta.\end{aligned}$$

Making the substitution $r' := \rho(r)$, we get $r \, dr = \eta M_\delta (1 - r'^2)^{\eta-1} r' \, dr'$, so the previous inequality yields

$$\begin{aligned}
&\int_{D\setminus D_\delta} |f\circ\varphi(z)|^p \, dA_\lambda \\
&\leq \frac{\lambda+1}{\pi}\int_0^{2\pi}\int_{\rho(\delta)}^1 |f(r'e^{i\theta})|^p \eta M_\delta(1-r'^2)^{\eta-1} r' M_\delta^\lambda (1-r'^2)^{\lambda\eta} dr' d\theta \\
&= M_\delta^{\lambda+1}\int_{D\setminus D_{\rho(\delta)}} |f(z)|^p dA_\gamma.
\end{aligned}$$

Now we conclude from the above inequalities that

$$\begin{aligned}
||C_\varphi||_e^p &\lesssim \lim_{\delta\to1} \ M_\delta^{\lambda+1} \sup_{||f||_{\gamma,p}\leq 1} \int_{D\setminus D_{\rho(\delta)}} |f(z)|^p dA_\gamma \\
&= \left(\limsup_{|z|\to1} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^\eta}\right)^{\lambda+1}.
\end{aligned}$$

□

Put $\alpha > -1,\ \lambda = \beta > -1$ and $\eta = \frac{\alpha+1}{\beta+1}$. According to the previous theorem, if

$$\sup_{z\in D} \frac{1-|z|^2}{(1-|\varphi(z)|^2)^{\frac{\alpha+1}{\beta+1}}} < \infty,$$

then $C_\varphi : A^p_\alpha \to A^p_\beta$ is bounded and

$$||C_\varphi||^p_e \lesssim \limsup_{|z|\to 1} \frac{(1-|z|^2)^{\beta+1}}{(1-|\varphi(z)|^2)^{\alpha+1}},$$

which means that the upper estimate in our first main result is proved.

The following example shows that the finiteness of the supremum in the assumption of our main result is not equivalent with the boundedness of the composition operator.

Example 1. Let $\alpha > \beta > -1,\ p > 1$ and $\varphi(z) = 1 - (1-z)^b$, where $b = (\beta+2)/(\alpha+2)$. It is easy to show using the Carleson measure theorem that the operator C_φ maps A^p_α into A^p_β non-compactly. However,

$$\sup_{z\in D} \frac{(1-|z|^2)^{\beta+1}}{(1-|\varphi(z)|^2)^{\alpha+1}} = \infty.$$

Note that in the case $\alpha < \beta$, the limit supremums in our main result are vanish, since for any analytic selfmap φ of D,

$$\sup_{z\in D} \frac{1-|z|^2}{1-|\varphi(z)|^2} < \infty.$$

In a recent paper ***Ueki*** has presented an argument that is claimed to show that the essential norm of a bounded composition operator $C_\varphi : A^2_\alpha \to A^2_\beta$, $\beta \geq \alpha$, is equivalent to the lower bound obtained in our main result for $p = 2$. But this argument seems to be in error. More precisely, if $C_\varphi : A^2_\alpha \to A^2_\beta$ is bounded, then $A^2_\beta \subset A^2_{2(\beta+1)}$ so C_φ can also be considered as a bounded operator from A^2_α into $A^2_{2(\beta+1)}$. Now in Ueki's paper, it is stated that the adjoint operators of these two composition operators coincide when restricted to functions from the unit ball of A^2_β. These adjoint operators depend on the duality, and they are explicitly given by

$$(C_\varphi)^* f(z) = (1+\beta) \int_D \frac{f(w)(1-|z|^2)^\beta}{(1-z\overline{\varphi(w)})^{\alpha+2}} dA(w), \ f \in A^2_\beta,$$

and

$$(C_\varphi)^t f(z) = (3+2\beta) \int_D \frac{f(w)(1-|z|^2)^{2(\beta+1)}}{(1-z\overline{\varphi(w)})^{\alpha+2}} dA(w), \ f \in A^2_{2(\beta+1)},$$

respectively.

Now, for example, if we pick the normalized kernel functions

$$k_z^\beta(w) := \frac{(1-|z|^2)^{\frac{2+\beta}{2}}}{(1-\overline{z}w)^{2+\beta}} \in A_\beta^2,$$

then it is easily verified that

$$\lim_{|z|\to 1} ||C_\varphi^*(k_z^\beta)||_{\alpha,2}^2 = \lim_{|z|\to 1} \frac{(1-|z|^2)^{\beta+2}}{(1-|\varphi(z)|^2)^{\alpha+2}},$$

which give the lower bound of the essential norm of $C_\varphi : A_\alpha^2 \to A_\beta^2$, but on the other hand, $\lim_{|z|\to 1} ||C_\varphi^t(k_z^\beta)||_{\alpha,2}^2 = 0$.

Norm-attaining weighted composition operators

The question of norm-attaining composition operators was first explicitely studied by ***Hammond*** (2003) in the setting of the Hardy space H^2. In 2005 Harmond also studied norm-attaining composition operators on the classical Dirichlet space for a certain special class of self-maps. Motivated by this, ***Martín*** (2009) characterized norm-attaining composition operators acting on the classical Bloch space $\mathcal{B}$ as well as on the little Bloch space $\mathcal{B}_0$.

I will now discuss how her results can be generalized to the setting of weighted composition operators acting on weighted Banach spaces of analytic functions.

Recall that a bounded linear operator T on a Banach space X attains its norm on X if there exists a function $f \in X$ with norm 1 such that $\|T\| = \|Tf\|$. We say that a function f with these properties is an ***extremal function*** for the norm of T.

James proved that a Banach space X is reflexive if and only if every compact operator on X is norm-attaining.

The so called ***associated weight*** is an important tool to handle problems in the setting of weighted Banach spaces of analytic functions. For a weight v the associated weight $\tilde{v}$ is defined by

$$\tilde{v}(z) := (\sup\{|f(z)|;\ f \in H_v^\infty, \|f\|_v \leq 1\})^{-1} = (\|\delta_z\|_v)^{-1},\ z \in D,$$

where δ_z denotes the point evaluation of z.

The ***classical Bloch space***

$$\mathcal{B} = \{f \in H\,(D) : f(0) = 0,\ \|f\|_{\mathcal{B}} = \sup_{z \in D}(1 - |z|^2)|f'(z)| < \infty\},$$

and the ***little Bloch space***

$$\mathcal{B}_0 = \{f \in \mathcal{B} : \lim_{|z| \to 1}(1 - |z|^2)|f'(z)| = 0\}$$

are Banach spaces endowed with the norm $|| \cdot ||_{\mathcal{B}}$.

For a bounded weighted composition operator $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$, the norm of $C_{\phi,\psi}$ is given by

$$\|C_{\phi,\psi}\|_{H_v^\infty \to H_w^\infty} = \sup_{z \in D} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))},$$

and, if v is a ”typical” weight, then it is known the essential norm of such an operator is given by

$$\|C_{\phi,\psi}\|_{e,H_v^\infty \to H_w^\infty} = \lim_{r \to 1} \sup_{|\phi(z)| > r} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))}.$$

The following result can be obtained:

Theorem 3. *If the weighted composition operator* $C_{\phi,\psi} : \mathcal{B} \longrightarrow \mathcal{B}$ *is bounded, then*

$$||C_{\phi,\psi}||_{\mathcal{B}\to\mathcal{B}} \simeq \max\left\{ ||C_{\phi,\psi'}||_{H^\infty_{(\log \frac{1}{1-|z|^2})^{-1}} \to H^\infty_{1-|z|^2}}, ||C_{\phi,\phi'\psi}||_{H^\infty_{1-|z|^2} \to H^\infty_{1-|z|^2}} \right\}.$$

Martín's result that every (bounded) composition operator C_ϕ on the Bloch space is norm-attaining is based on the following result due to ***Gorkin-Mortini*** (2004):

Lemma 2. *Let* $(z_n) \subset D$ *be a sequence such that* $|z_n| \longrightarrow 1$*, when* $n \longrightarrow \infty$*. Then there exists an infinite Blaschke product* B *whose zeros are elements of the sequence* (z_n) *and such that* $|B'(z_n)|(1-|z_n|^2) \longrightarrow 1$*, when* $n \longrightarrow \infty$*.*

Our second main result is the following:

Theorem 4. *Every bounded weighted composition operator* $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$ *is norm-attaining.*

For the proof we need the following result.

Lemma 3. *Let* $(z_m) \subset D$ *be a sequence such that* $|z_m| \to 1$*, when* $m \to \infty$*. Then there is a subsequence* (z_n) *of* (z_m) *and a function* $g \in H_v^\infty$*,* $||g||_v \leq 1$*, such that* $|g(z_n)|\tilde{v}(z_n) \to 1$*, when* $n \to \infty$*. In particular, there is a* $h \in \mathcal{B}$*,* $||h||_{\mathcal{B}} \leq 1$*, with* $|h'(z_n)|(1 - |z_n|^2) \to 1$*, when* $n \to \infty$*.*

For the second statement of the above lemma we apply the first result with the weight $v_1(z) = 1 - |z|^2$. Consider now the bounded operators $S : \mathcal{B} \to H_{v_1}^\infty,\ S(h) = h'$ and $S^{-1} : H_{v_1}^\infty \to \mathcal{B},\ (S^{-1}h)(z) = \int_0^z h(\xi)d\xi$. Clearly S, S^{-1} are isometric onto maps.

Then, let $h := S^{-1}(g) \in \mathcal{B}$, so $||h||_{\mathcal{B}} = ||g||_{v_1} \leq 1$ and $h' = g$.

Consider a composition operator $C_\phi : \mathcal{B} \to \mathcal{B}$. Then we can use Theorem 4 to find an extremal function $g \in H^\infty_{v_1}$ for the norm of $C_{\phi,\phi'} : H^\infty_{v_1} \to H^\infty_{v_1}$, and therefore

$$||C_\phi||_{\mathcal{B}\to\mathcal{B}} = ||C_{\phi,\phi'}||_{H^\infty_{v_1}\to H^\infty_{v_1}} = ||SC_\phi S^{-1}||_{H^\infty_{v_1}\to H^\infty_{v_1}} = ||(SC_\phi S^{-1})g||_{v_1}.$$

Now using that S is an isometry, it follows that $h := S^{-1}(g) \in \mathcal{B}$ is an extremal function for the norm of C$_\phi : \mathcal{B} \to \mathcal{B}$. Thus we obtain Martin's Theorem:

Corollary 1. *Every composition operator $C_\phi : \mathcal{B} \to \mathcal{B}$ is norm-attaining.*

The following result is a consequence of the above theorems 3 and 4.

Corollary 2. *For every bounded weighted composition operator* $C_{\phi,\psi} : \mathcal{B} \to \mathcal{B}$*, there are norm-one functions* $f \in H^\infty_{(\log \frac{1}{1-|z|^2})^{-1}}$*,* $g \in H^\infty_{1-|z|^2}$ *such that*

$$||C_{\phi,\psi}||_{\mathcal{B}\to\mathcal{B}} \simeq \max\left\{||C_{\phi,\psi'} f||_{1-|z|^2}, ||C_{\phi,\phi'\psi} g||_{1-|z|^2}\right\}.$$

Finally let me mention the corresponding results for the small spaces H_v^0:

Theorem 5. *The bounded operator* $C_{\phi,\psi} : H_v^0 \to H_w^0$ *is norm-attaining if and only if we can find a point* $b \in D$ *and a sequence* $(z_n)_n \subset D$ *such that* $\lim_{n\to\infty} \phi(z_n) = b$ *and*

$$\sup_{z\in D} \frac{|\psi(z)|w(z)}{\tilde{v}(\phi(z))} = \lim_{n\to\infty} \frac{|\psi(z_n)|w(z_n)}{\tilde{v}(\phi(z_n))}.$$

Moreover, if the bounded operator $C_{\phi,\psi}$ *is norm-attaining on* H_v^0*, then the function* $f \in H_v^0$ *is norm-attaining if and only if there is a point* b *as in the first part and such that additionally the condition*

$$f(b)\tilde{v}(b) = 1 \text{ is satisfied.}$$

Corollary 3. *Let* v *be a typical weight and* $C_{\phi,\psi}$ *be a bounded operator on* H_v^0*. If* $\|C_{\phi,\psi}\|_{e,H_v^0\to H_v^0} < \|C_{\phi,\psi}\|_{H_v^0\to H_v^0}$*, then the operator is norm-attaining.*

The next is the proof of Lemma 1:

Proof. From the assumption and the maximum modulus principle of analytic functions we deduce, that φ maps the disk $\overline{D_r}$ into the disk $\overline{D_\rho}$.

By a classical theorem of complex analysis, there exist a function $H : \overline{D_\rho} \to \mathbb{R}$ harmonic in D_ρ such that $H(\rho e^{i\theta}) = |f(\rho e^{i\theta})|^p$ for every $0 \leq \theta < 2\pi$.

By the assumption, $|f|^p$ is subharmonic in D, so $|f|^p \leq H$ in $\overline{D_\rho}$. We also assumed that φ is analytic and $\varphi(\overline{D_r}) \subset \overline{D_\rho}$. Hence the function $H \circ \varphi$ is harmonic in D_r and $|f \circ \varphi|^p \leq H \circ \varphi$ in $\overline{D_r}$. Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} |(f \circ \varphi)(re^{i\theta})|^p \, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} (H \circ \varphi)(re^{i\theta}) \, d\theta = (H \circ \varphi)(0)$$

$$= H(0) = \frac{1}{2\pi}\int_0^{2\pi} H(\rho e^{i\theta}) \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta$$

by the meanvalue property of harmonic functions. □