Banach spaces with the Daugavet property

(joint papers with Vladimir Kadets, Nigel Kalton, Miguel Martín, Enrique Sánchez et al.)

Dirk Werner
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Each compact linear operator $T\colon C[0,1] \to C[0,1]$ satisfies

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$C[0,1]$, $L_1[0,1]$, $L_\infty[0,1]$, $A(\mathbb{D})$, H^∞, $\mathrm{Lip}(K)$ ($K \subset \mathbb{R}^d$ convex), type II von Neumann algebras and their preduals, ...

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Counterexamples

c_0, ℓ_1, ℓ_∞, $L_p(\mu)$ for $1 < p < \infty$, $\mathrm{Lip}(K)$ ($K \subset \mathbb{R}^d$ compact and not convex), type I von Neumann algebras and their preduals, ...

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Definition

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for all operators $T\colon X \to X$ of the form $T(x) = x_0^*(x)\,x_0$.

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- For all $\|x_0\| = 1$ and $\varepsilon > 0$, the convex hull of $\{z \in B_X\colon \|z - x_0\| \geq 2 - \varepsilon\}$ is dense in B_X.

Weak compactness

Proposition

If X has the Daugavet property, then $\|\mathrm{Id} + T\| = 1 + \|T\|$ for all weakly compact operators T.

T is weakly compact if the closure of $T(B_X)$ is weakly compact, i.e., compact for the weak topology.

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If X has the Daugavet property, then $\|\mathrm{Id} + T\| = 1 + \|T\|$ for all strong Radon-Nikodym operators T.

T is a strong Radon-Nikodym operator if the closure of $T(B_X)$ has the Radon-Nikodym property.

ℓ_1-subspaces

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Example

In $C[0, 2\pi]$, the functions $t \mapsto \sin(2^n t)$ span a copy of ℓ_1.

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Theorem

If X has the Daugavet property, then $\|\mathrm{Id} + T\| = 1 + \|T\|$ for all ℓ_1-singular operators T.

T is called ℓ_1-singular if *no* restriction of T to any copy of ℓ_1 is an (into-) isomorphism, i.e., bounded below.

Unconditional bases

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Even more, a separable Banach space with the Daugavet property does not even embed into a space with an unconditional basis.

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Indeed, for these T the sum $T + S$ is narrow whenever S is narrow (although in general the set of narrow operators is not a vector space).

Rich subspaces

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Definition

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- If $(X/U)^*$ is separable, then U is rich.

Example

There is a Banach space X with the Daugavet property that fails to contain a copy of $L_1[0,1]$. Indeed, X can be chosen to be of the form $L_1[0,1]/(U_1 \oplus_1 U_2 \oplus_1 \cdots)$ with each U_n isomorphic to ℓ_1.

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Question 2: What if U only has the RNP?

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Corollary

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The almost Daugavet property

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X has the almost Daugavet property if there exists a norming subspace $Y \subset X^*$ such that

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The following are equivalent for a separable Banach space X:

- X has the almost Daugavet property.
- There exists a sequence (e_n^*) in X^* that is isometrically equivalent to the unit vector basis of ℓ_1 and such that $\mathrm{lin}\{e_k^* : k \geq n\}$ is norming for every n.

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Some properties

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$$n(X) = n(X^*) \text{ ???}$$

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$n(X) = 1$ if and only if $\max_{\pm} \|\mathrm{Id} \pm T\| = 1 + \|T\|$ for all $T\colon X \to X$.

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Lemma

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Note:
Daugavet property $\not\Rightarrow$ $n(X) = 1$ (e.g. $X = C([0,1], \mathbb{R}^2)$);
$n(X) = 1 \not\Rightarrow$ Daugavet property (e.g. $X = c_0$).

Definition

A (real) Banach space X is called lush if for all $\|x_0\| = 1$, $\|y_0\| = 1$ and $\varepsilon > 0$ there exists an ε-slice S containing x_0 such that $\operatorname{dist}(y_0, \operatorname{co}(S \cup -S)) \le \varepsilon$.

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Proposition

Every lush space has numerical index 1.

The solution of the duality problem

Theorem

Let $X = \{f \in C[0,2]\colon f(0) + f(1) + f(2) = 0\}$. Then $n(X) = 1$, but $n(X^*) \le 1/2$.

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Theorem

There is a real Banach space with $n(X)=1$, but $n(X^*)=0$.

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All contributions are welcome!