About convergence and Cauchyness in fuzzy metric spaces

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Definition

(George and Veeramani [1].) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;

(GV2) $M(x, y, t) = 1$ if and only if $x = y$;

(GV3) $M(x, y, t) = M(y, x, t)$;

(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

If axioms (GV1), (GV2) and (GV5) are replaced by

(KM1) $M(x, y, 0) = 0$

(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$

(KM5) $M(x, y, \_) : [0, \infty[\to [0, 1]$ is left continuous

respectively, we obtain the concept of KM-fuzzy metric space.

Definition (George and Veeramani [1].)

Let (X, d) be a metric space and let M_d a fuzzy set on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space and M_d is called the standard fuzzy metric induced by d.

Definitions (George and Veeramani [1].)

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be M-Cauchy, or simply Cauchy, if for each $\epsilon \in]0, 1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$, or equivalently, if $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$.

Let (X, d) be a metric space and suppose X endowed with the topology induced by d.

$$\text{A sequence } \{x_n\} \text{ in } X \text{ is convergent to } x_0 \text{ if and only if } \lim_n d(x_0, x_n) = 0 \quad (1)$$

and

$$\{x_n\} \text{ is Cauchy } \text{ if and only if } \lim_{n,m} d(x_m, x_n) = 0 \quad (2)$$

Comparing (1) and (2) one can say, formally, that (2) is deduced, in a natural way, from (1) replacing x_0 by x_m and tacking double limit or *vice-versa* ((1) is obtained from (2) replacing x_m by x_0).

It is well-known that if d is an ultrametric (non-Archimedean) metric on X then

$$\{x_n\} \text{ is Cauchy } \text{ if and only if } \lim_n d(x_n, x_{n+1}) = 0 \quad (3)$$

Definition (Gregori and Miñana [5].)

Suppose it is given a stronger (weaker, respectively) concept than Cauchy sequence, say *s*-Cauchy sequence (*w*-Cauchy sequence, respectively). A concept of convergence, say *s*-convergence (*w*-convergence, respectively), is said to be compatible with *s*-Cauchy (*w*-Cauchy, respectively), and *vice-versa*, if the diagram of implications below on the left (on the right, respectively) is fulfilled

$$\begin{array}{ccc} s-convergence & \rightarrow & convergence \\ \downarrow & & \downarrow \\ s-Cauchy & \rightarrow & Cauchy \end{array} \qquad \begin{array}{ccc} convergence & \rightarrow & w-convergence \\ \downarrow & & \downarrow \\ Cauchy & \rightarrow & w-Cauchy \end{array}$$

and there is not any other implication, in general, among these concepts.

A sequence $\{x_n\}$ is G-Cauchy if $\lim_n M(x_{n+p}, x_n, t) = 1$ for each $t > 0$ and $p \in \mathbb{N}$. A fuzzy metric space in which every G-Cauchy sequence is convergent is called G-complete.

In this way Grabiec for a certain class of KM-fuzzy metric spaces was able to state an elegant fuzzy version of the Banach Contraction Principle: Every fuzzy contractive self-mapping on a G-complete fuzzy metric space X admits a unique fixed point ([3], Theorem 5).

Now, an inconvenience of the concept of G-Cauchy sequence is that a compact fuzzy metric space is not necessarily G-complete as it was proved in [15], Example 3.7.

As it was observed by Mihet in [11], a sequence $\{x_n\}$ is G-Cauchy if and only if $\lim_n M(x_n, x_{n+1}, t) = 1$ for all $t > 0$ (compare with (3)).

So, if one tries to define a concept of G-convergent sequence to x_0 in X, imitating the classical case and attending to the definition of G-Cauchy sequence, it is obtained the following concept:

A sequence $\{x_n\}$ is G-convergent to x_0 if $\lim_n M(x_0, x_{n+1}, t) = 1$ for each $t > 0$, which are equivalent to $\lim_n M(x_0, x_n, t) = 1$ for each $t > 0$, i.e., it is the usual concept of convergence to x_0.

D. Mihet introduced in [10] the following weaker concept than convergence of sequences: A sequence $\{x_n\}$ is p-convergent (to x_0) if $\lim_n M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$.
The authors in [4] showed that every p-convergent sequence in a fuzzy metric space $(X, M, *)$ is convergent if and only if for each $t > 0$ the family $\{B(x, r, t) : r \in]0, 1[\}$ is a local base at x, for each $x \in X$. These spaces were called principal fuzzy metric spaces.
A sequence $\{x_n\}$ is p-Cauchy if there exists $t_0 > 0$ such that for each $\epsilon \in]0, 1[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t_0) > 1 - \epsilon$ for all $n, m \geq n_0$, or equivalently, $\lim_{n,m} M(x_n, x_m, t_0) = 1$ for some $t_0 > 0$.
From the results obtained in [4] one concludes that the next diagram is fulfilled

$$\begin{array}{ccc} convergence & \rightarrow & p-convergence \\ \downarrow & & \downarrow \\ Cauchy & \rightarrow & p-Cauchy \end{array}$$

In order to establish a relationship between the theory of complete fuzzy metric spaces and domain theory the authors introduced in [13] in fuzzy metric spaces the following stronger concept than Cauchy sequence, which we denote *std*-Cauchy, as follows: A sequence $\{x_n\}$ is *std*-Cauchy if for each $\epsilon \in]0,1[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > \frac{t}{t+\epsilon}$ for all $n, m \geq n_0$ and for all $t > 0$.

Then, in a natural way and imitating the classical case the authors gave in [12] the following concept: A sequence $\{x_n\}$ is *std*-convergent to x_0 if for each $\epsilon \in]0,1[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_0, t) > \frac{t}{t+\epsilon}$ for all $n \geq n_0$ and for all $t > 0$.

The authors have shown in [5] that there exist *std*-convergent sequences, which are not *std*-Cauchy. Then, in the sense of [5], the concept of *std*-convergence is not compatible with *std*-Cauchy.

A sequence $\{x_n\}$ in X is s-convergent to $x_0 \in X$ if $\lim_n M(x_n, x_0, \frac{1}{n}) = 1$. A fuzzy metric space in which every convergent sequence is s-convergent is called s-fuzzy metric space. s-fuzzy metric spaces are characterized in [7] as follow: M is an s-fuzzy metric if and only if $\bigcap_{t>0} B(x, r, t)$ is a neighbourhood of x, for all $x \in X$ and for all $r \in]0, 1[$, or equivalently, $\{\bigcap_{t>0} B(x, r, t) : r \in]0, 1[\}$ is a local base at x, for each $x \in X$.
If $(X, M, *)$ is a fuzzy metric space where $N(x, y) = \bigwedge_{t>0} M(x, y, t) > 0$ for all $x, y \in X$, then N is a stationary fuzzy metric on X. In [7] it is proved that $\tau_N = \tau_M$ if and only if M is an s-fuzzy metric.

Imitating the classical case and seeing the above definition it is natural to give the next definition: A sequence $\{x_n\}$ is s-Cauchy if $\lim_{n,m} M(x_n, x_m, \frac{1}{n}) = 1$. In this case we have the next proposition.

Proposition

Every s-Cauchy sequence is Cauchy.

Proof.

Suppose that $\{x_n\}$ is s-Cauchy. Let $t > 0$ and take $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < t$. Then we have that $M(x_n, x_m, t) \geq M(x_n, x_m, \frac{1}{n_0}) \geq M(x_n, x_m, \frac{1}{n})$ for all $n \geq n_0$ and all $m \in \mathbb{N}$. Then $\lim_n M(x_n, x_m, t) = 1$. □

Unfortunately, as in the case of *std*-convergence, an s-convergent sequence is not necessarily s-Cauchy, as shows the next example.

Example

Consider $(\mathbb{R}, M_d, \cdot)$. We will see that the sequence $\{x_n\}$, given by $x_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$, is s-convergent to 0 but it is not s-Cauchy.

$$\lim_n M_d\left(x_n, 0, \frac{1}{n}\right) = \lim_n \frac{\frac{1}{n}}{\frac{1}{n} + \frac{1}{n^2}} = \lim_n \frac{1}{1 + \frac{1}{n}} = 1.$$

Now, we will see that $\{x_n\}$ is not s-Cauchy. Contrary, we suppose that $\{x_n\}$ is s-Cauchy, that is

$$\lim_{n,m} M_d\left(x_n, x_m, \frac{1}{n}\right) = \lim_{n,m} \frac{\frac{1}{n}}{\frac{1}{n} + \left|\frac{1}{n^2} - \frac{1}{m^2}\right|} = 1.$$

Now, for values of n and m large, if we take $m = \sqrt{n}$ we have that

$$\lim_{n,\sqrt{n}} \frac{\frac{1}{n}}{\frac{1}{n} + \left|\frac{1}{n^2} - \frac{1}{(\sqrt{n})^2}\right|} = \lim_n \frac{\frac{1}{n}}{\frac{1}{n} + \frac{1}{n} - \frac{1}{n^2}} = \lim_n \frac{1}{2 - \frac{1}{n}} = \frac{1}{2}.$$

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