Extreme points in compact convex sets in asymmetric normed spaces

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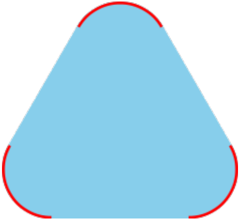
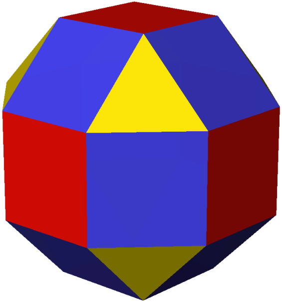
Motivation

Let X be a vector space and $K \subset X$. Recall that a point $x \in K$ is an extreme point of K if and only if $x = \frac{1}{2}(y + z)$ with $y, z \in K$ implies that $x = y = z$.





Motivation

(Krein-Milman)

Every compact convex subset of a locally convex (Hausdorff) space is the closure of the convex hull of its extreme points.

In particular, each compact convex subset of a locally convex space has at least an extreme point.

(Carathéodory)

Each finite dimensional compact convex set in a locally convex (Hausdorff) space is the convex hull of its extreme points.

Motivation

In general, Krein-Milman theorem is not longer valid in asymmetric normed spaces.

Is it possible to describe the geometric structure of compact convex sets in an asymmetric normed space?

Asymmetric normed space

Definition

Let X be a (real)-vector space. An asymmetric norm or quasi-norm in X is a function $q : X \to [0, \infty)$ satisfying

1. $q(x) = 0 = q(-x)$ if and only if $x = 0$.
2. $q(\lambda x) = \lambda q(x)$ for each $\lambda \geq 0$ and $x \in X$.
3. $q(x + y) \leq q(x) + q(y)$ for every $x, y \in X$.

The pair (X, q) is usually called an asymmetric normed space.

Example

Let $u : \mathbb{R} \to [0, \infty)$ defined by

$$u(x) = \text{máx}\{x, 0\} = x^{+}.$$

The pair $(\mathbb{R}, u)$ is an asymmetric normed space.

Example

In $\mathbb{R}^n$ define $q_\infty : \mathbb{R}^n \to [0,\infty)$ by the rule

$$q_\infty((x_1, x_2, \ldots, x_n)) = \text{máx}\{x_1^+, x_2^+, \ldots, x_n^+\}.$$

$(\mathbb{R}^n, q_\infty)$ is an example of an asymmetric normed space.

Let (X, q) be an asymmetric normed space. For every $x \in X$ and $\varepsilon > 0$, define

$$B_q(x, \varepsilon) = \{y \in X \mid q(y - x) < \varepsilon\}$$

The family $\left\{B_q(x, \varepsilon) \mid x \in X,\ \varepsilon > 0\right\}$ is a base for a topology τ_q in (X, q). We will refer to this topology as the "asymmetric topology"

This topology is always a T_0 topology. But, in general, it is not Hausdorff.

Example

In $(\mathbb{R}^n, q_\infty)$, a basic open set has the form

$$(-\infty, a_1) \times (-\infty, a_2) \times \cdots \times (-\infty, a_n)$$

with $a_1, \ldots, a_n \in \mathbb{R}$.

For any asymmetric normed space (X, q) there is always an associated symmetric norm

$$q^s(x) = \text{máx}\{q(x), q(-x)\}$$

- The topology generated by the norm q^s is very helpful. We denote this topology by τ_{q^s} and we will call it the "symmetric topology"
- $\tau_q \subset \tau_{q^s}$.

Let (X, q) be an asymmetric normed space. We denote by $\theta(0) = \{x \in X \mid q(x) = 0\}$.

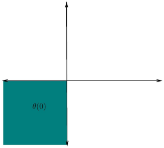
$\theta(0)$ is a convex cone.

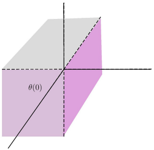
Example

- In $(\mathbb{R}, u)$, $\theta(0) = (-\infty, 0]$.

- In $(\mathbb{R}^n, q_\infty)$

  $\theta(0) = (-\infty, 0]^n$





Properties of $\theta(0)$:

- *(X,q) is T_1 if and only if $\theta(0)=\{0\}$.*
- *If U is q-open, then $U=U+\theta(0)$.*
- *K is q-compact if and only if $K+\theta(0)$ is q-compact.*

The main problem while studying compact sets in asymmetric normed spaces is that, in general, compact sets are not closed.

Krein-Milman theorem

In general, Krein-Milman theorem is not valid in asymmetric normed spaces.

Example

The set $(-1, 0]$ is a u-compact convex set in $(\mathbb{R}, u)$. The only extreme point of $(-1, 0]$ is 0, and the closure of 0 coincides with the interval $[0, \infty)$

(Cobzaç, 2004)

Let (X, q) be an asymmetric normed space such that the topology τ_q is Hausdorff. Then any nonempty q-compact convex subset of X is the q-closed convex hull of the set of its extreme points.

Theorem

Let K be a q-compact convex subset of an asymmetric normed space (X, q) with the property that $K + \theta(0)$ is q^s-locally compact. Then K has at least one extreme point. In particular, if $K + \theta(0)$ has finite dimension, then K has at least one extreme point.

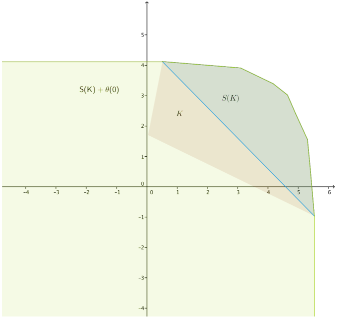
In contrast with the normed case, let us observe that the previous theorem is the best we can say about extreme points in q-compact convex sets. For instance, in any asymmetric normed space (X, q), the set $\theta(x) = x + \theta(0)$ is a q-compact convex set for whom its only extreme point is x itself.

For any convex set $K \subset X$, let us denote by $S(K)$ the convex hull of all the extreme points of K.

Theorem (NJ and E. Sánchez)

Let (X, q) be a finite dimensional asymmetric normed space and K a q-compact convex subset of X. Then

$$S(K) \subset K \subset S(K) + \theta(0) = K + \theta(0).$$

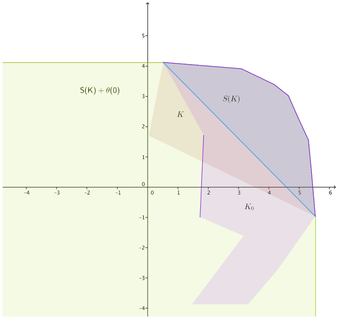


Corollary

Let K be a q-compact convex subset in a finite dimensional asymmetric normed space (X, q). If $K_0 \subset X$ is any subset satisfying

$$S(K) \subset K_0 \subset S(K) + \theta(0)$$

then K_0 is q-compact.



Definition

Let (X, q) be an asymmetric normed space. A set $K_0 \subset X$ is *q-strongly compact* (or simply, strongly compact) iff there exists $K \subset X$ such that K is q^s-compact and

$$K \subset K_0 \subset K + \theta(0)$$

In an asymmetric normed space (X, q), a q^s-compact set is q-compact.

In $(\mathbb{R}, u)$ every compact convex set is strongly compact and is an interval of the form $[a, b]$ or $(a, b]$ or $(-\infty, b]$ where $a, b \in \mathbb{R}$.

In $(\mathbb{R}, u)$ every compact set is strongly compact!

(N. J. and E. A. Sánchez - Pérez)

Let q be an asymmetric lattice norm in $\mathbb{R}^2$. Then every q-compact convex set in $\mathbb{R}^2$ is strongly compact.

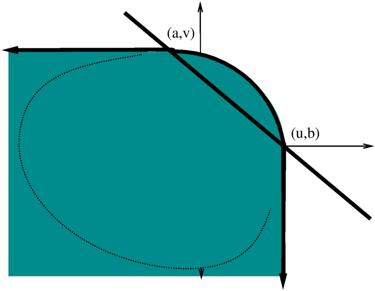
Let q an asymmetric lattice norm in $\mathbb{R}^2$ induced by the coordinatewise order. Consider a q-compact convex set $K \subset \mathbb{R}^2$. Let $P_1, P_2 : \mathbb{R}^2 \to \mathbb{R}$ be the projections in the first and second coordinates, respectively, and define:

$$u := \sup\{P_1((x,y)) : (x,y) \in K\}$$

$$v := \sup\{P_2((x,y)) : (x,y) \in K\}$$

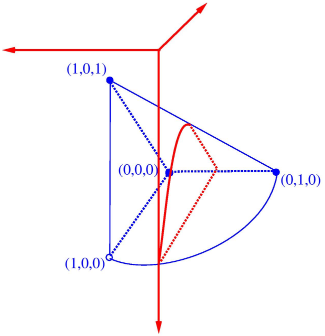
$$a := \sup\{P_1((x,v)) : (x,v) \in K\}$$

$$b := \sup\{P_2((u,y)) : (u,y) \in K\}$$



In this case $K_0 = S(K + \theta(0))$

In dimension 3, we can find q-compact convex sets which are not strongly compact.



Consider the asymmetric normed lattice $(\mathbb{R}^3, q_\infty)$.
Define $K = \text{conv}(A \cup \{(0,0,0),(1,0,1)\}) \setminus \{(1,0,0)\}$ where A is the set

$$A = \{(x_1, x_2, 0) \mid x_1^2 + x_2^2 = 1,\ x_1, x_2 \geq 0\}.$$

Then K is a compact convex set which is not strongly compact.

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Thank you very much for your attention!