PROPERTIES OF OPERATORS FROM BANACH FUNCTION SPACES THAT EXTEND TO THEIR OPTIMAL DOMAINS

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Notation

- (Ω, Σ, μ) positive finite measure space.
- $X(\mu)$ be a σ-o.c. Banach function space, E Banach space.
- $T : X(\mu) \to E$ continuous linear operator.
- $T(\chi_A) = m_T(A)$, $A \in \Sigma$ σ-additive.
- $L^1(m_T)$ space of m_T-integrable functions.
- $I_{m_T} : f \to \int_\Omega f dm_T$ is the integration operator.
- T is μ-determined: the m_T-null ($\mathcal{N}(m_T)$) μ-null ($\mathcal{N}(\mu)$) sets are the same. $J_T : X(\mu) \hookrightarrow L^1(m_T)$ inclusion.

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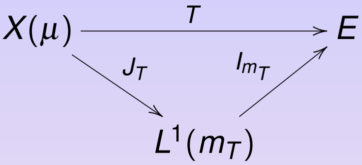
1. 

2. Every $f \in X(\mu)$ is m_T-integrable and $T(f\chi_A) = \int_A f dm_T \; A \in \Sigma$.
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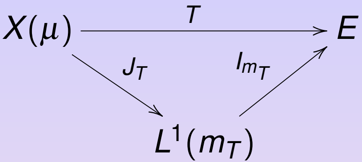
4. J_T is injective whenever $\mathcal{N}(\mu) = \mathcal{N}(m_T)$.In this case, $L^1(m_T)$ is a B.f.s. into which $X(\mu)$ is continuously embedded via the map J_T with $T = I_{m_T} \circ J_T$ is the unique continuous linear extension of T to $L^1(m_T)$. That is, the diagram is commutative.

1. 

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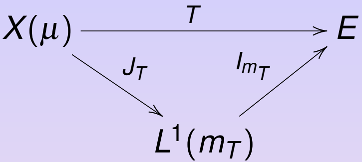
1. 

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1. $$\begin{array}{ccc} X(\mu) & \xrightarrow{\quad T \quad} & E \\ & {\scriptstyle J_T} \searrow \quad \nearrow {\scriptstyle I_{m_T}} & \\ & L^1(m_T) & \end{array}$$
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The Curbera-Ricker Optimal Domain Theorem asserts that $L^1(m_T)$ is the largest σ-order continuous B.f.s. over (Ω, Σ, μ) into which $X(\mu)$ is continuously embedded and to which T admits an E-valued continuous linear extension.

G.P. Curbera and W.J. Ricker, Optimal domains for kernel operators via interpolation. Math. Nachr. 244 (2002), 47-63.

If we consider $T : X(\mu) \to E$ a compact operator, a natural question raised by E.A. Sánchez Pérez is whether or not T admits a maximal compact linear extension.

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Theorem

Suppose that $X(\mu)$ is a σ-order continuous B.f.s. based on a positive, finite measure space (Ω, Σ, μ) and that $T : X(\mu) \to E$ is a μ-determined, compact linear operator. Then T admits a maximal compact linear extension if and only if the integration operator $I_{m_T} : L^1(m_T) \to E$ is compact.

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Theorem

Suppose that $X(\mu)$ is a σ-order continuous B.f.s. based on a positive, finite measure space (Ω, Σ, μ) and that $T : X(\mu) \to E$ is a μ-determined, weakly compact linear operator. Then T admits a maximal weakly compact linear extension if and only if the integration operator $I_{m_T} : L^1(m_T) \to E$ is weakly compact.

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Volterra integral operator

$$(Vf)(t) := \int_0^t f(u)du, \quad t \in [0,1] \quad f \in L^1([0,1])$$

$$\begin{array}{ccc} L^1([0,1]) & \xrightarrow{\quad V \quad} & L^1([0,1]) \\ & {\scriptstyle J_T} \searrow \qquad \nearrow {\scriptstyle I_{m_V} = L^1((1-t)dt)} & \\ & L^1(m_V) & \end{array}$$

μ Lebesgue measure on the Borel σ-algebra $\mathscr{B}([0,1])$. The operator V admits a natural extension $\tilde{V}$ to the space of all functions $f \in L^0(\mu)$ such that f is Lebesgue integrable over $[0,u]$ for every $u \in [0,1]$ and the function $u \mapsto \int_0^u f(t)dt$ belongs to $L^1([0,1])$ in which case $\tilde{V}(f)(u) = \int_0^u f(t)dt$: Via Fubini's Theorem $L^1(m_V) = L^1((1-t)dt)$ is the maximal σ-order continuous domain.

V is compact and I_{m_V} is not compact and not weakly compact.

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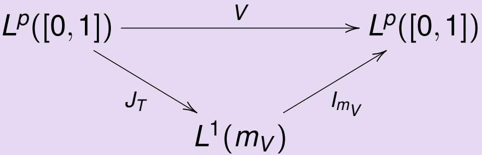

μ Lebesgue measure on the Borel σ-algebra $\mathscr{B}([0,1])$. Its maximal continuous linear extension is also non-compact but, obviously weakly compact due to reflexivity of the codomain space $L^p([0,1])$,
V is compact and I_{m_V} is not compact and but it is weakly compact.

Motivation

Which properties of operators can they be extend to their optimal domain?

$L^0(\mu)$

We denote by $L^0(\mu)$ the space of all measurable real functions on Ω, where functions which are equal μ-a.e. are identified. The space $L^0(\mu)$ will be endowed with the μ-a.e. pointwise order, that is, $f \leq g$ if and only if $f \leq g$ μ-a.e. Then, $L^0(\mu)$ is a vector lattice.

Banach function space

Let $X(\mu)$ be a Banach function space related to μ we mean a Banach space $X(\mu) \subset L^0(\mu)$ satisfying that if $|f| \leq |g|$ with $f \in L^0(\mu)$ and $g \in X(\mu)$ then $f \in X(\mu)$ and $\|f\| \leq \|g\|$.

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AM-compact operator

An operator T from a B.f.s. $X(\mu)$ into a Banach lattice E is said to be *AM*-compact if it transforms order bounded subsets of $X(\mu)$ into precompact subsets of E.

Theorem

Let (Ω, Σ, μ) be a positive finite measure space. Let $X(\mu)$ be a σ-order continuous B.f.s. over (Ω, Σ, μ) and E is a Banach space. A μ-determined AM-compact operator $T : X(\mu) \to E$ admits a maximal *AM*-compact linear extension if and only if the integration operator $I_{m_T} : L^1(m_T) \to E$ is *AM*-compact.

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Sketch of the proof

1. $T : X(\mu) \to E$ μ-determined operator,
2. $1 < p < \infty$, $g \in L^q(m_T)$, $g \geq c\chi_\Omega$, for some $c > 0$, $1/p + 1/q = 1$.
3. $g \cdot L^p(m_T)$ is a σ-order continuous B.f.s. based on (Ω, Σ, μ)

Lemma

The operator $I_{m_T}^{(g,p)}$ is *AM*-compact if and only if $\mathbf{R}(m_T) = \{m_T(A) : A \in \Sigma\} = \{T(\chi_A) : A \in \Sigma\}$ of the vector measure $m_T : \Sigma \to E$ is relatively compact.

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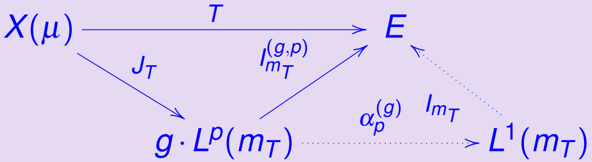

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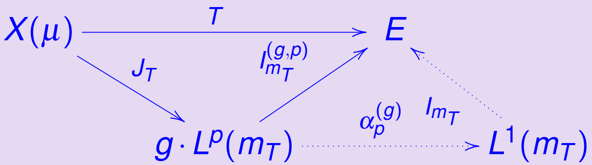

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1. $\Leftarrow$ is obvious.
2. Conversely suppose that I_{m_T} is not *AM*-compact. $Y(\mu)$ over (Ω, Σ, μ) such that $X(\mu)$ is continuously embedded in $Y(\mu)$ and for which $T_{Y(\mu)} : Y(\mu) \to E$ is an *AM*-compact linear extension. The continuity of $T_{Y(\mu)}$ implies that $Y(\mu)$ is continuously embedded into $L^1(m_T)_\mu$. Since I_{m_T} is not *AM*-compact and $T_{Y(\mu)}$ is *AM*-compact we have that $Y(\mu) \subsetneq L^1(m_T)_\mu$.

$$Z(\mu) := Y(\mu) + g \cdot L^p(m_T)$$

$$\|f\|_{Z(\mu)} := \inf(\|\phi\|_{Y(\mu)} + \|\psi\|_{g \cdot L^p(m_T)})$$

Let B be an order bounded subset in $Z(\mu)$ and the proof finish showing that the restriction I_{m_T} to $Z(\mu)$ provides a proper AM-compact linear extension of $T_{Y(\mu)}$.

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Dunford-Pettis operator

A linear operator $T : X \to Y$ between two Banach spaces X, Y is called Dunford-Pettis if T sends weakly null sequences from X to norm null sequences in Y. These operators are often also called completely continuous.

Remark

In general for a Dunford-Pettis operator T from a B.f.s $X(\mu)$ into a Banach space E, the subset $\{T(\chi_A) : A \in \Sigma\}$ is a relatively compact set in E. However the converse is false.

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Let $T : X(\mu) \to E$ be a μ-determined Dunford-Pettis linear operator where $X(\mu)$ is a σ-order continuous B.f.s. based on (Ω, Σ, μ). Then T admits a maximal Dunford-Pettis linear extension if and only if the integration operator I_{m_T} is Dunford-Pettis.

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Corollary

Let $X(\mu)$ be a B.f.s. and let E be a Banach space. If $T : X(\mu) \to E$ is a μ-determined Dunford-Pettis operator then T is narrow and the integration operator $I_{m_T} : L^1(m_T)_\mu \to E$ is the maximal μ-narrow linear extension.

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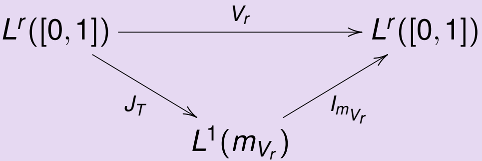
Volterra integral operator

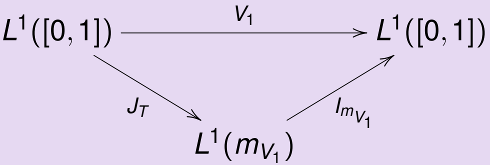
$$(V_r f)(t) := \int_0^t f(u)du, \quad t \in [0,1], \quad f \in L^r([0,1]), \quad 1 \le r \le \infty$$



$m_{V_r} : \mathscr{B}([0,1]) \to L^r([0,1])$. $m_{V_r}(A) = V_r(\chi_A)$ for $A \in \mathscr{B}([\imath,\infty])$.
For each $1 \le r \le \infty$ the operator V_r is compact.
Therefore V_r is Dunford-Pettis.
But $I_{m_{V_r}}$ is not Dunford-Pettis operator.

Volterra integral operator

$$(V_1 f)(t) := \int_0^t f(u)du, \quad t \in [0,1], \quad f \in L^1([0,1])$$



$m_{V_1} : \mathscr{B}([0,1]) \to L^1([0,1])$. $m_{V_1}(A) = V_1(\chi_A)$ for $A \in \mathscr{B}([\imath, \infty])$.
The operator V_1 is compact.
Therefore V_1 is AM-compact.
But $I_{m_{V_1}}$ is not AM-compact operator.

Sign function

We define $\Sigma^+ := \{A \in \Sigma : \mu(A) > 0\}$. A function $f \in L^0(\mu)$ is called a sign if it takes values in the set $\{-1, 0, 1\}$, and a sign on $A \in \Sigma$ if it is a sign with $\mathrm{supp} f = A$. We say that a sign f is of mean zero if $\int_\Omega f d\mu = 0$.

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Narrow operator

An operator $T : X(\mu) \to E$ is called narrow, where $X(\mu)$ is a B.f.s. and E a Banach space, if for each $A \in \Sigma^+$ and each $\varepsilon > 0$ there exists a mean zero sign f on A such that $\|T(f)\| < \varepsilon$.

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Narrow operator

Let $X(\mu)$ be a non-atomic order complete vector lattice, a linear operator $T : X(\mu) \to E$ is narrow, where E is a Banach space, if for each $f \in X(\mu)^+$ and each $\varepsilon > 0$ there exists some $g \in X(\mu)$ such that $|g| = f$ and $\|T(g)\| < \varepsilon$

Theorem

Let $X(\mu)$ be a σ-order continuous B.f.s. and let E be a Banach space. Let $T : X(\mu) \to E$ be a μ-determined linear operator. Then T is narrow if and only if the integration operator $I_{m_T} : L^1(m_T)_\mu \to E$ is narrow.

Corollary

Let $T : L^1(\mu) \to E$ be a μ-determined continuous linear operator. Let E be a Banach space with the Radon-Nikodým property. Then the integration operator I_{m_T} is narrow and it is the maximal narrow extension.

Sign-embedding

An injective linear operator $T : X(\mu) \to E$ is called sign-embedding if for some $\delta > 0$ and every sign function $f \in X(\mu)$ then

$$\|T(f)\|_E \geq \delta \|f\|_{X(\mu)}$$

Corollary

Let E be a separable Banach space and $1 \leq p < \infty$. Let $T : L^p(\mu) \to E$ be a μ-determined narrow operator. Then I_{m_T} is not sign-embed in E.

Remark

Let $X(\mu)$ be a σ-order continuous B.f.s over a non atomic finite measure space $((0,1),\Sigma,\mu)$ and let $T : X(\mu) \to E$ be a μ-determined linear operator. Then the range of every E-valued measure m_T has convex closure then every bounded operator T is narrow.

Corollary

Let $X(\mu)$ be a σ-order continuous B.f.s.over a non atomic measure space (Ω). If the range $\mathbf{R}(m_T)$ of every E-valued measure m_T has convex closure being T a μ-determined linear operator then every integrator operator I_{m_T} is the maximal narrow linear extension.

Corollary

Let $1 \leq p < \infty$. If $T : L^p(\mu) \to E$ is a μ-determined absolutely summing operator then T is narrow and the integration operator $I_{m_T} : L^1(m_T)_\mu \to E$ is the maximal narrow linear extension.

corollary

Let $T : L^p(\mu) \to L^r(\mu)$ be a μ-determined linear operator, where $1 \leq p < 2$ and $p < r < \infty$, then:

1. T is narrow.
2. $I_{m_T} : L^1(m_T) \to L^r(\mu)$ is narrow.

corollary

Let $m : \Sigma \to L^r(\mu)$ be a countably vector measure such that exists a continuous inclusion $i : L^p(\mu) \hookrightarrow L^1(m)$ and $1 \leq p < 2$, $p < r < \infty$ then the integration operator $I_m : L^1(m) \to L^r(\mu)$ is narrow.

corollary

Let $T : L^p(\mu) \to c_0$ be a μ-determined linear operator, where $1 \leq p < \infty$:

1. T is narrow.
2. $I_{m_T} : L^1(m_T) \to c_0$ is narrow.

corollary

Let $m : \Sigma \to L^r(\mu)$ be a countably vector measure such that exists a continuous inclusion $i : L^p(\mu) \hookrightarrow L^1(m)$ and $1 \leq p < 2$, $p < r < \infty$ then the integration operator $I_m : L^1(m) \to L^r(\mu)$ is narrow.

corollary

Let $m : \Sigma \to L^r(\mu)$ be a countably vector measure such that exists a continuous inclusion $i : L^p(\mu) \hookrightarrow L^1(m)$ and $1 \leq p < 2$, $p < r < \infty$ then the integration operator $I_m : L^1(m) \to L^r(\mu)$ is narrow.

THANK YOU FOR YOUR ATTENTION