PROPERTIES OF OPERATORS FROM BANACH FUNCTION SPACES THAT EXTEND TO THEIR OPTIMAL DOMAINS

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Notation

- $(\Omega, \Sigma, \mu)$ positive finite measure space.
- $X(\mu)$ be a $\sigma$-o.c. Banach function space, $E$ Banach space.
- $T : X(\mu) \to E$ continuous linear operator.
- $T(\chi_A) = m_T(A)$, $A \in \Sigma$ $\sigma$-additive.
- $L^1(m_T)$ space of $m_T$-integrable functions.
- $l_{m_T} : f \to \int_{\Omega} f dm_T$ is the integration operator.
- $T$ is $\mu$-determined: the $m_T$-null ($\mathcal{N}(m_T)$) $\mu$-null ($\mathcal{N}(\mu)$) sets are the same. $J_T : X(\mu) \hookrightarrow L^1(m_T)$ inclusion.
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1. Every $f \in X(\mu)$ is $m_T$-integrable and $T(f \chi_A) = \int_A f dm_T$ $A \in \Sigma$.
2. $\mathcal{N}(\mu) \subseteq \mathcal{N}(m_T)$. The linear map $J_T$ is well defined and is continuous with $\|J_T\| = \|T\|$.
3. $J_T$ is injective whenever $\mathcal{N}(\mu) = \mathcal{N}(m_T)$. In this case, $L^1(m_T)$ is a B.f.s. into which $X(\mu)$ is continuously embedded via the map $J_T$ with $T = I_{m_T} \circ J_T$ is the unique continuous linear extension of $T$ to $L^1(m_T)$. That is, the diagram is commutative.
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The Curbera-Ricker Optimal Domain Theorem asserts that $L^1(m_T)$ is the largest $\sigma$-order continuous B.f.s. over $(\Omega, \Sigma, \mu)$ into which $X(\mu)$ is continuously embedded and to which $T$ admits an $E$-valued continuous linear extension.

If we consider $T : X(\mu) \to E$ a compact operator, a natural question raised by E.A. Sánchez Pérez is whether or not $T$ admits a maximal compact linear extension.

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**Theorem**

Suppose that $X(\mu)$ is a $\sigma$-order continuous B.f.s. based on a positive, finite measure space $(\Omega, \Sigma, \mu)$ and that $T : X(\mu) \to E$ is a $\mu$-determined, compact linear operator. Then $T$ admits a maximal compact linear extension if and only if the integration operator $I_{m_T} : L^1(m_T) \to E$ is compact.

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**Theorem**

Suppose that $X(\mu)$ is a $\sigma$-order continuous B.f.s. based on a positive, finite measure space $(\Omega, \Sigma, \mu)$ and that $T : X(\mu) \to E$ is a $\mu$-determined, weakly compact linear operator. Then $T$ admits a maximal weakly compact linear extension if and only if the integration operator $I_{m_T} : L^1(m_T) \to E$ is weakly compact.

Volterra integral operator

\[(Vf)(t) := \int_0^t f(u) \, du, \quad t \in [0, 1] \quad f \in L^1([0, 1])\]

\[\begin{array}{ccc}
L^1([0, 1]) & \xrightarrow{V} & L^1([0, 1]) \\
J_T & \xrightarrow{\mu} & L^1((1-t) \, dt) \\
L^1(m_V) & \xrightarrow{I_{m_V}} & L^1(m_V) = L^1((1-t) \, dt)
\end{array}\]

\(\mu\) Lebesgue measure on the Borel \(\sigma\)-algebra \(\mathcal{B}([0, 1])\). The operator \(V\) admits a natural extension \(\tilde{V}\) to the space of all functions \(f \in L^0(\mu)\) such that \(f\) is Lebesgue integrable over \([0, u]\) for every \(u \in [0, 1]\) and the function \(u \mapsto \int_0^u f(t) \, dt\) belongs to \(L^1([0, 1])\) in which case \(\tilde{V}(f)(u) = \int_0^u f(t) \, dt\): Via Fubini’s Theorem \(L^1(m_V) = L^1((1-t) \, dt)\) is the maximal \(\sigma\)-order continuous domain. \(V\) is compact and \(I_{m_V}\) is not compact and not weakly compact.
Volterra integral operator

\[(Vf)(t) := \int_0^t f(u)du, \quad t \in [0, 1] \quad f \in L^1([0, 1])\]

\[L^p([0, 1]) \xrightarrow{V} L^p([0, 1])\]

\[L^1(m_V) \xrightarrow{J_T} V \xrightarrow{I_{m_V}} \]

\(\mu\) Lebesgue measure on the Borel \(\sigma\)-algebra \(\mathcal{B}([0, 1])\). Its maximal continuous linear extension is also non-compact but, obviously weakly compact due to reflexivity of the codomain space \(L^p([0, 1])\), 

\(V\) is compact and \(I_{m_V}\) is not compact and but it is weakly compact.
Motivation

Which properties of operators can they be extend to their optimal domain?
$L^0(\mu)$

We denote by $L^0(\mu)$ the space of all measurable real functions on $\Omega$, where functions which are equal $\mu$-a.e. are identified. The space $L^0(\mu)$ will be endowed with the $\mu$-a.e. pointwise order, that is, $f \leq g$ if and only if $f \leq g$ $\mu$-a.e. Then, $L^0(\mu)$ is a vector lattice.

**Banach function space**

Let $X(\mu)$ be a Banach function space related to $\mu$ we mean a Banach space $X(\mu) \subset L^0(\mu)$ satisfying that if $|f| \leq |g|$ with $f \in L^0(\mu)$ and $g \in X(\mu)$ then $f \in X(\mu)$ and $\|f\| \leq \|g\|$. 
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**AM-compact operator**

An operator $T$ from a B.f.s. $X(\mu)$ into a Banach lattice $E$ is said to be *AM-compact* if it transforms order bounded subsets of $X(\mu)$ into precompact subsets of $E$.

**Theorem**

Let $(\Omega, \Sigma, \mu)$ be a positive finite measure space. Let $X(\mu)$ be a $\sigma$-order continuous B.f.s. over $(\Omega, \Sigma, \mu)$ and $E$ is a Banach space. A $\mu$-determined AM-compact operator $T : X(\mu) \to E$ admits a maximal AM-compact linear extension if and only if the integration operator $I_{m_T} : L^1(m_T) \to E$ is AM-compact.
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Let $(\Omega, \Sigma, \mu)$ be a positive finite measure space. Let $X(\mu)$ be a $\sigma$-order continuous B.f.s. over $(\Omega, \Sigma, \mu)$ and $E$ is a Banach space. A $\mu$-determined AM-compact operator $T : X(\mu) \to E$ admits a maximal AM-compact linear extension if and only if the integration operator $I_{m_T} : L^1(m_T) \to E$ is AM-compact.
Sketch of the proof

1. $T : X(\mu) \to E$ $\mu$-determined operator,
2. $1 < p < \infty$, $g \in L^q(m_T)$, $g \geq c\chi_\Omega$, for some $c > 0$, $1/p + 1/q = 1$.
3. $g \cdot L^p(m_T)$ is a $\sigma$-order continuous B.f.s. based on $(\Omega, \Sigma, \mu)$

\[ \begin{align*}
X(\mu) & \xrightarrow{T} E \\
& \xrightarrow{J_T} L^p(m_T) \\
& \xrightarrow{\alpha_p} L^1(m_T)
\end{align*} \]

Lemma

The operator $I_{m_T}^{(g,p)}$ is AM-compact if and only if $R(m_T) = \{m_T(A) : A \in \Sigma\} = \{T(\chi_A) : A \in \Sigma\}$ of the vector measure $m_T : \Sigma \to E$ is relatively compact.
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$X(\mu)$ $\xrightarrow{T}$ $E$

$J_T$ $\xrightarrow{T^{(g,p)}}$ $E$

$g \cdot L^p(m_T)$ $\xrightarrow{\alpha_p^{(g)}}$ $L^1(m_T)$

Lemma

The operator $I^{(g,p)}_{m_T}$ is AM-compact if and only if $R(m_T) = \{ m_T(A) : A \in \Sigma \} = \{ T(\chi_A) : A \in \Sigma \}$ of the vector measure $m_T : \Sigma \to E$ is relatively compact.
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Sketch of the proof

1. $\iff$ is obvious.

2. Conversely suppose that $I_{m_T}$ is not $AM$-compact. $Y(\mu)$ over $(\Omega, \Sigma, \mu)$ such that $X(\mu)$ is continuously embedded in $Y(\mu)$ and for which $T_{Y(\mu)} : Y(\mu) \to E$ is an $AM$-compact linear extension. The continuity of $T_{Y(\mu)}$ implies that $Y(\mu)$ is continuously embedded into $L^1(m_T)$. Since $I_{m_T}$ is not $AM$-compact and $T_{Y(\mu)}$ is $AM$-compact we have that $Y(\mu) \subsetneq L^1(m_T)$.

\[
Z(\mu) := Y(\mu) + g \cdot L^p(m_T)
\]

\[
\|f\|_{Z(\mu)} := \inf(\|\phi\|_{Y(\mu)} + \|\psi\|_{g \cdot L^p(m_T)})
\]

Let $B$ be an order bounded subset in $Z(\mu)$ and the proof finish showing that the restriction $I_{m_T}$ to $Z(\mu)$ provides a proper $AM$-compact linear extension of $T_{Y(\mu)}$. 
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E. Jiménez Fernández (adjimfer@ugr.es)
**Dunford-Pettis operator**

A linear operator $T : X \rightarrow Y$ between two Banach spaces $X$, $Y$ is called **Dunford-Pettis** if $T$ sends weakly null sequences from $X$ to norm null sequences in $Y$. These operators are often also called completely continuous.

**Remark**

In general for a Dunford-Pettis operator $T$ from a B.f.s $X(\mu)$ into a Banach space $E$, the subset $\{ T(\chi_A) : A \in \Sigma \}$ is a relatively compact set in $E$. However the converse is false.
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Theorem

Let $T : X(\mu) \rightarrow E$ be a $\mu$-determined Dunford-Pettis linear operator where $X(\mu)$ is a $\sigma$-order continuous B.f.s. based on $(\Omega, \Sigma, \mu)$. Then $T$ admits a maximal Dunford-Pettis linear extension if and only if the integration operator $I_{mT}$ is Dunford-Pettis.

Remark

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Dunford-Pettis operator

A linear operator \( T : X \to Y \) between two Banach spaces \( X, Y \) is called \textbf{Dunford-Pettis} if \( T \) sends weakly null sequences from \( X \) to norm null sequences in \( Y \). These operators are often also called completely continuous.

Corollary

Let \( X(\mu) \) be a B.f.s. and let \( E \) be a Banach space. If \( T : X(\mu) \to E \) is a \( \mu \)-determined Dunford-Pettis operator then \( T \) is narrow and the integration operator \( I_{m_T} : L^1(m_T)_\mu \to E \) is the maximal \( \mu \)-narrow linear extension.

Remark

In general for a Dunford-Pettis operator \( T \) from a B.f.s \( X(\mu) \) into a Banach space \( E \), the subset \( \{ T(\chi_A) : A \in \Sigma \} \) is a relatively compact set in \( E \). However the converse is false.
Volterra integral operator

\[(V_rf)(t) := \int_0^t f(u)du, \quad t \in [0,1], \quad f \in L^r([0,1]), \quad 1 \leq r \leq \infty\]

For each \(1 \leq r \leq \infty\) the operator \(V_r\) is compact.

Therefore \(V_r\) is Dunford-Pettis.

But \(I_{mV_r}\) is not Dunford-Pettis operator.

\[m_{V_r} : \mathcal{B}([0,1]) \to L^r([0,1]). \quad m_{V_r}(A) = V_r(\chi_A) \text{ for } A \in \mathcal{B}([0,1])\]
**Volterra integral operator**

\[(V_1 f)(t) := \int_0^t f(u) \, du, \quad t \in [0, 1], \quad f \in L^1([0, 1])\]

\[L^1([0, 1]) \xrightarrow{V_1} L^1([0, 1]) \xrightarrow{J_T} L^1(m_{V_1}) \xrightarrow{l_{m_{V_1}}} L^1([0, 1])\]

\[m_{V_1} : B([0, 1]) \to L^1([0, 1]). \quad m_{V_1}(A) = V_1(\chi_A) \text{ for } A \in B([t, \infty]).\]

The operator \(V_1\) is compact. Therefore \(V_1\) is AM-compact.

But \(l_{m_{V_1}}\) is not AM-compact operator.
Sign function

We define $\Sigma^+ := \{ A \in \Sigma : \mu(A) > 0 \}$. A function $f \in L^0(\mu)$ is called a sign if it takes values in the set $\{-1, 0, 1\}$, and a sign on $A \in \Sigma$ if it is a sign with $\text{supp}f = A$. We say that a sign $f$ is of mean zero if $\int_{\Omega} f d\mu = 0$. 
Sign function

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Narrow operator

An operator $T : X(\mu) \to E$ is called narrow, where $X(\mu)$ is a B.f.s. and $E$ a Banach space, if for each $A \in \Sigma^+$ and each $\varepsilon > 0$ there exists a mean zero sign $f$ on $A$ such that $\| T(f) \| < \varepsilon$. 
Sign function

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Narrow operator

Let $X(\mu)$ be a non-atomic order complete vector lattice, a linear operator $T : X(\mu) \to E$ is narrow, where $E$ is a Banach space, if for each $f \in X(\mu)^+$ and each $\varepsilon > 0$ there exists some $g \in X(\mu)$ such that $|g| = f$ and $\|T(g)\| < \varepsilon$. 

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Theorem

Let $X(\mu)$ be a $\sigma$-order continuous B.f.s. and let $E$ be a Banach space. Let $T : X(\mu) \to E$ be a $\mu$-determined linear operator. Then $T$ is narrow if and only if the integration operator $I_{m_T} : L^1(m_T)_\mu \to E$ is narrow.
Corollary

Let $T : L^1(\mu) \rightarrow E$ be a $\mu$-determined continuous linear operator. Let $E$ be a Banach space with the Radon-Nikodým property. Then the integration operator $I_{mT}$ is narrow and it is the maximal narrow extension.
Sign-embedding

An injective linear operator $T : X(\mu) \rightarrow E$ is called sign-embedding if for some $\delta > 0$ and every sign function $f \in X(\mu)$ then

$$\| T(f) \|_E \geq \delta \| f \|_{X(\mu)}$$

Corollary

Let $E$ be a separable Banach space and $1 \leq p < \infty$. Let $T : L^p(\mu) \rightarrow E$ be a $\mu$-determined narrow operator. Then $l_{ImT}$ is not sign-embed in $E$. 
Remark

Let $X(\mu)$ be a $\sigma$-order continuous B.f.s over a non atomic finite measure space $((0,1), \Sigma, \mu)$ and let $T : X(\mu) \to E$ be a $\mu$-determined linear operator. Then the range of every $E$-valued measure $m_T$ has convex closure then every bounded operator $T$ is narrow.
Corollary

Let $X(\mu)$ be a $\sigma$-order continuous B.f.s. over a non atomic measure space $(\Omega)$. If the range $R(m_T)$ of every $E$-valued measure $m_T$ has convex closure being $T$ a $\mu$-determined linear operator then every integrator operator $I_{m_T}$ is the maximal narrow linear extension.

Corollary

Let $1 \leq p < \infty$. If $T : L^p(\mu) \to E$ is a $\mu$-determined absolutely summing operator then $T$ is narrow and the integration operator $I_{m_T} : L^1(m_T)_\mu \to E$ is the maximal narrow linear extension.
corollary

Let $T : L^p(\mu) \to L^r(\mu)$ be a $\mu$-determined linear operator, where $1 \leq p < 2$ and $p < r < \infty$, then:

1. $T$ is narrow.
2. $I_{m_T} : L^1(m_T) \to L^r(\mu)$ is narrow.

corollary

Let $m : \Sigma \to L^r(\mu)$ be a countably vector measure such that exists a continuous inclusion $i : L^p(\mu) \hookrightarrow L^1(m)$ and $1 \leq p < 2$, $p < r < \infty$ then the integration operator $I_m : L^1(m) \to L^r(\mu)$ is narrow.
corollary

Let $T : L^p(\mu) \to c_0$ be a $\mu$-determined linear operator, where $1 \leq p < \infty$:

1. $T$ is narrow.
2. $I_{m_T} : L^1(m_T) \to c_0$ is narrow.

corollary

Let $m : \Sigma \to L^r(\mu)$ be a countably vector measure such that exists a continuous inclusion $i : L^p(\mu) \leftrightarrow L^1(m)$ and $1 \leq p < 2$, $p < r < \infty$ then the integration operator $I_m : L^1(m) \to L^r(\mu)$ is narrow.
**corollary**

Let \( m : \Sigma \to L^r(\mu) \) be a countably vector measure such that exists a continuous inclusion \( i : L^p(\mu) \leftrightarrow L^1(m) \) and \( 1 \leq p < 2, \ p < r < \infty \) then the integration operator \( I_m : L^1(m) \to L^r(\mu) \) is narrow.
THANK YOU FOR YOUR ATTENTION