Multivariate polynomials and polynomial inequalities via random processes and interpolation

Mieczysław Mastyło

Adam Mickiewicz University in Poznań

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Outline

- Let $\mathcal{P}(\mathbb{K}^n)$ be the space of all polynomials P on $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, i.e.,

$$P(z) = \sum_{\alpha} c_\alpha(P) z^\alpha \,, \quad z = (z_1, \dots, z_n) \in \mathbb{K}^n \,.$$

Here the sum is taken over finitely many multi indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $z^\alpha := z_1^{\alpha_1} \dots z_n^{\alpha_n}$ denotes the αth monomial. As usual, $|\alpha| = \sum_{j=1}^n \alpha_j$ and we call

$$\deg(P) := \max\{|\alpha| \,;\, c_\alpha(P) \neq 0\}$$

the total degree of P. If $m = \deg(P)$ and all monomial coefficients $c_\alpha = c_\alpha(P) = 0$ for all $|\alpha| < m$, then P is said to be m-homogeneous.

- For every polynomial $P \in \mathcal{P}(\mathbb{K}^n)$ and $(u_1, \dots, u_n) \in \mathbb{K}^n$ the degree of the one variable polynomial

$$P_j(\cdot) = P(u_1, \ldots, u_{j-1}, \cdot, u_{j+1}, \ldots, u_n)$$

equals $\deg(P_j) := \max\left\{\alpha_j \,;\, c_\alpha(P) \neq 0\right\}$.

- The Mahler measure of a polynomial $P\colon \mathbb{C}^n \to \mathbb{C}$ is given by

$$M(P) := \exp \int_{\mathbb{T}^n} \log |P|\, dz = \lim_{p \to 0+} \left(\int_{\mathbb{T}^n} |P|^p \, dz \right)^{1/p},$$

where $dz = dz_1 \ldots dz_n$ stands for the normalized Lebesgue measure on the n-torus $\mathbb{T}^n$. Thus $M(P)$ is the geometric mean of P over the n-torus $\mathbb{T}^n$ (we define $M(0) = 0$).

- D. H. Lehmer [Ann. of Math. (1933)] proved that, if $P(z) = \sum_{k=0}^{m} a_k z^k$ for $z \in \mathbb{C}$ with $a_m \neq 0$ and zeros $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$, then

$$M(P) = |a_m| \prod_{|\alpha_i| \geqslant 1} |\alpha_i|.$$

- the multiplicativity property of the Mahler measure M: For two polynomials P and Q on $\mathbb{C}^n$,

$$M(PQ) = M(P)M(Q).$$

- Arestov (1980) For every polynomial P in one complex variable and of degree $\deg(P) \leqslant m$, and every $0 < p < \infty$, we have

$$\|P\|_{L^p(\mathbb{T})} \leqslant \Lambda(p, m)\, M(P),$$

where

$$\Lambda(p, m) := \Big(\int_{\mathbb{T}} |1 + z|^{mp}\, dz \Big)^{1/p} = 2^m \pi^{-1/2p} \left(\frac{\Gamma(2^{-1}(mp + 1))}{\Gamma(2^{-1}(mp + 2))} \right)^{1/p}.$$

By definition, we have that $\|P\|_{L^p(\mathbb{T})} = \Lambda(p, m)$ for the polynomial $P(z) = (1 + z)^m$, $z \in \mathbb{C}$, and moreover $M(P) = 1$, which implies that Arestov result is sharp.

- The study of L^p-norms of polynomials has a rich history. Recently, there has been a considerable interest in the behaviour of the constants of Khintchine-Kahane type equivalences $\|\cdot\|_{L^p} \approx \|\cdot\|_{L^q}$ in the case when L^p-spaces are considered on arbitrary unit-volume convex bodies K in $\mathbb{R}^n$ (works by Gromov-Milman (1983/84) and Bourgain (1991).

- Bobkov (2000) If μ is an arbitrary log-concave probability measure μ on $\mathbb{R}^n$, then, for all $1 \leqslant p < \infty$ and all $P \in \mathcal{P}(\mathbb{R}^n)$, we have

$$\|P\|_{L^p(\mu)} \leqslant \left(p^{\frac{22}{\ln 2}}\right)^{\deg(P)} \|P\|_{L^1(\mu)} .$$

- Nikol'skii (1951, 1975) For every $P \in \mathcal{P}(\mathbb{C}^n)$ and $0 < q < p < \infty$,

$$\|P\|_{L^q(\mathbb{T}^n)} \leqslant 2^n \left(\prod_{j=1}^{n} \deg(P_j) \right)^{1/q-1/p} \|P\|_{L^p(\mathbb{T}^n)} .$$

- Bayart (2002) For for each homogeneous polynomial $P \in \mathcal{P}(\mathbb{C}^n)$ and $0 < p < q < \infty$ we have,

$$\|P\|_{L^q(\mathbb{T}^n)} \leqslant \left(\sqrt{\frac{q}{p}} \right)^{\deg(P)} \|P\|_{L^p(\mathbb{T}^n)} .$$

L^p-norms versus Mahler's measure for polynomials (A. Defant & M. M.)

Theorem

For every $P \in \mathcal{P}(\mathbb{C}^n)$ *and every* $0 < p < \infty$

$$\|P\|_{L^p(\mathbb{T}^n)} \leqslant \left(\prod_{j=1}^{n} \Lambda(p, d_j) \right) M(P)\,,$$

where $d_j = \deg(P_j)$ *for each* $1 \leqslant j \leqslant n$*. Moreover, this inequality is sharp since for the polynomial* $P(z_1, ..., z_n) = P_1(z_1) \cdots P_n(z_n)$, $(z_1, ..., z_n) \in \mathbb{C}^n$ *with* $P_j(z) = (1+z)^{d_j}$, $z \in \mathbb{C}$ *and* $d_j \in \mathbb{N}$, $1 \leqslant j \leqslant n$, *we have that*

$$\|P\|_{L^p(\mathbb{T}^n)} = \prod_{j=1}^{n} \Lambda(p, d_j), \quad M(P) = 1.$$

Corollary

For every $P \in \mathcal{P}(\mathbb{C}^n)$ *with* $m = \deg(P)$, *and every* $0 < p < \infty$ *we have*

$$\|P\|_{L^p(\mathbb{T}^n)} \leqslant \Lambda(p, m)^n \, M(P).$$

- Bourgain (1991) (answering affirmatively a question by Milman). There exist universal constants $t_0 > 0$ and $c \in (0,1)$ such that, for every convex set $K \subset \mathbb{R}^n$ of volume one and every $P \in \mathcal{P}(\mathbb{R}^n)$, the following distribution inequality holds

$$\mu_K\{|P| \geqslant t\,\|P\|_1\} \leqslant \exp(-t^{c/n}), \quad t \geqslant t_0,$$

where μ_K is the Lebesgue measure on K, and $\|f\|_1$ is the L^1-norm of P with respect to μ_K.

- Remark. The above inequality may be rewritten in terms of the Orlicz space $L_{\psi_\alpha}(K)$ generated by the convex function $\psi_\alpha(t) = \exp(t^\alpha) - 1$, $t \geqslant 0$ as follows:

$$\|P\|_{L_{\psi_\alpha}(K)} \leqslant C^n \|P\|_1, \quad P \in \mathcal{P}(\mathbb{R}^n)\,,$$

where $\alpha = c/n$ and $C > 0$ is some absolute constant.

Theorem

For every $P \in \mathcal{P}(\mathbb{C}^n)$ *with* $m = \deg(P)$ *we have*

$$\|P\|_{L_{\psi_{1/m}}(\mathbb{T}^n)} \leqslant (2^n(e-1))^m \, M(P)\,.$$

Lemma

For every $P, Q \in \mathcal{P}(\mathbb{C}^n)$ *with* $m = \deg(P)$ *and* $k = \deg(Q)$, *and every* $0 < p < \infty$ *we have*

$$\|PQ\|_{L^p(\mathbb{T}^n)} \geqslant \big(\Lambda(p,m)\,\Lambda(p,k)\big)^{-n}\|P\|_{L^p(\mathbb{T}^n)}\,\|Q\|_{L^p(\mathbb{T}^n)}\,.$$

In particular, if $P(z) = \sum_\alpha c_\alpha(P) z^\alpha$ *and* $Q(z) = \sum_\alpha c_\alpha(Q) z^\alpha$, *then*

$$\Big(\sum_\alpha |c_\alpha(PQ)|^2\Big)^{1/2} \geqslant \binom{2m}{m}^{-n/2}\binom{2k}{k}^{-n/2}\Big(\sum_\alpha |c_\alpha(P)|^2\Big)^{1/2}\Big(\sum_\alpha |c_\alpha(Q)|^2\Big)^{1/2}.$$

- Mahler (1961) proved the following univariate "triangle inequality",

$$M(P+Q) \leqslant \kappa(m)\,\big(M(P)+M(Q)\big), \quad P,\,Q \in \mathcal{P}(\mathbb{C})$$

with the constant $\kappa(m) = 2^m$, where $\deg(P) = \deg(Q) = m$, and observed that it has applications in the theory of diophantine approximation.

- Duncan (1966) obtained the above inequality with the smaller constant

$$\kappa(m) = \binom{2m}{m}^{1/2} \approx (\pi m)^{-1/4}\,2^m\,.$$

- Arestov (1990) The following two-sided estimates hold:

$$\frac{1}{2} r^m \leqslant \kappa(m) \leqslant \frac{1}{2} R^m, \quad m \geqslant 6$$

where $R = \sqrt[6]{40} \approx 1,8493$ and $r = \exp(2G/\pi) \approx 1,7916$ with $G = \sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$.

The following multidimensional variant of Duncan's result is true.

Corollary

For $P,\ Q \in \mathcal{P}(\mathbb{C}^n)$ with $m = \deg(P) = \deg(Q)$ we have

$$M(P+Q) \leqslant \binom{2m}{m}^{n/2} (M(P) + M(Q)).$$

Theorem

Let E be a Lebesgue measurable subset of $\mathbb{T}^n$ with $\lambda_n(E) \geqslant \theta > 0$. Then the following interpolation inequality holds for any polynomial $P \in \mathcal{P}(\mathbb{C}^n)$ with $\deg(P) = m$ and any $0 < p < \infty$:

$$\|P\|_{L^p(\mathbb{T}^n)} \leqslant C(\theta)\,\Lambda(p,m)^n \left(\|P\|_{L^1(\mathbb{T}^n)}\right)^{1-\theta} \left(\|P\|_{L^1(E)}\right)^{\theta},$$

where $C(\theta) = \frac{1}{\theta^\theta(1-\theta)^{1-\theta}} \leqslant 2$ for every $0 < \theta < 1$, and $C(1) = 1$.

Kahane-Salem-Zygmund polynomial inequalities

- Kahane-Salem-Zygmund (J. P. Kahane, 1993) For each m, $n \in \mathbb{N}$ and every polynomial $P(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha$ on $\mathbb{C}^n$ we have

$$\int_\Omega \sup_{z \in \mathbb{D}^n} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha(\omega) c_\alpha z^\alpha \Big| \, d\mathbb{P}(\omega) \leqslant C \Big(n \log m \sum_{|\alpha|=m} |c_\alpha|^2 \Big)^{1/2},$$

where C is a positive constant that depends neither on n nor on m and $(\varepsilon_\alpha)_{|\alpha|=m}$ is a family of independent Bernoulli variables on a probability measure space $(\Omega, \mathcal{A}, \mathbb{P})$. Here $\mathbb{D}^n$ is the n-dimensional polydisk in $\mathbb{C}^n$

- Random polynomials with small supremum norms. For each positive integers m and n, there exists a choice of signs $(\varepsilon_\alpha)_{|\alpha|=m}$, $\varepsilon_\alpha = \pm 1$, such that

$$\sup_{z \in \mathbb{D}^n} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha \Big| \leqslant C n^{(m+1)/2} \sqrt{\log m},$$

where C is a positive real number which does not depend on m nor on n.

- The *k*-th entropy number $\varepsilon_k(S) := \varepsilon_k(S\colon X \to Y)$ is defined by

$$\varepsilon_k(S) = \inf\Big\{\varepsilon > 0;\ S(B_X) \subset \bigcup_{j=1}^{k}(y_j + \varepsilon B_Y) \ \text{for some}\ \ y_1, \ldots, y_k \in Y\Big\}.$$

  The n-th (dyadic) entropy number $e_n(S)$ of S is given by $e_n(S) := \varepsilon_{2^{n-1}}(S)$ for each $n \in \mathbb{N}$.

- Given a pseudo-metric (T, d), we denote by $N(T, d; \varepsilon)$ the entropy function associated with the pseudo-metric d on the set T for $\varepsilon > 0$, i.e.,

$$N(T, d; \varepsilon)$$

  is the smallest number of open balls of radius $\varepsilon > 0$ in the pseudo-metric d needed to cover the set T.

- Let Φ be a Young function on $\mathbb{R}_+ := [0, \infty)$, i.e., $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex increasing function such that $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. The entropy integral of (T, d) with respect to Φ is defined by

$$J_\Phi(T, d) = \int_0^{\Delta(T)} \Phi^{-1}\big(N(T, d; \varepsilon)\big)\, d\varepsilon,$$

  where $\Delta(T) = \sup_{s,t\in T} d(s, t)$ denotes the diameter of T.

- A basic result in the study of regularity of stochastic processes which is of fundamental importance states that if $(X_t)_{t\in T}$ is a stochastic process in the Orlicz space on a probability measure space $L_\Phi = L_\Phi(\Omega, \mathcal{A}, \mathbb{P})$ of bounded increments, i.e.,

$$\|X_s - X_t\|_{L_\Phi} \leqslant d(s, t), \quad s, t \in T,$$

  then we have

$$\mathbb{E}\Big(\sup_{s,t\in T} |X_s - X_t|\Big) \leqslant 8J_\Phi(T, d).$$

- For each $m,\ n \in \mathbb{N}$ the following index sets will be of special interest:

$$\mathcal{M}(m,n) = \{\mathbf{j} = (j_1, \ldots, j_m)\,;\, 1 \leqslant j_1, \ldots, j_m \leqslant n\} = \{1, \ldots, n\}^m,$$

$$\mathcal{J}(m,n) = \{\mathbf{j} = (j_1, \ldots, j_m) \in \mathcal{M}(m,n)\,;\, 1 \leqslant j_1 \leqslant j_2 \leqslant \ldots \leqslant j_m \leqslant n\},$$

$$\Lambda(m,n) = \{\alpha \in \mathbb{N}_0^n\,;\, |\alpha| = \alpha_1 + \ldots + \alpha_n = m\}.$$

- For $\mathbf{i}, \mathbf{j} \in \mathcal{M}(m,n)$, we write $\mathbf{i} \sim \mathbf{j}$ whenever there exists a permutation σ of $\{1, \ldots, m\}$ such that $(i_1, \ldots, i_m) = (j_{\sigma(1)}, \ldots, j_{\sigma(m)})$. Then, $\sim$ defines an equivalence relation on $\mathcal{M}(m,n)$. We denote by $[\mathbf{i}]$ the equivalence class of $\mathbf{i}$ and we put $|[\mathbf{i}]| := \mathrm{card}([\mathbf{i}])$.

- For every finite subset $\{x_1^*, \ldots, x_n^*\}$ in the dual X^* of a Banach space X and each $\mathbf{j} \in \mathcal{M}(m,n)$, an m-homogeneous polynomial $x_{\mathbf{j}}^* : X \to \mathbb{C}$ is defined by

$$x_{\mathbf{j}}^*(x) = x_{j_1}^*(x) \cdots x_{j_m}^*(x), \quad x \in X.$$

We note that $\mathbf{i} \sim \mathbf{j}$ implies that $x_{\mathbf{i}}^* = x_{\mathbf{j}}^*$, for each $\mathbf{i}, \mathbf{j} \in \mathcal{M}(m,n)$.

Definition Given a Young function Φ and positive integers $m \geqslant 2$ and $n \geqslant 1$, Banach sequence spaces $E = E(\mathbb{N})$ and $F = F(\mathcal{J}(m,n))$ are said to be polynomially Φ-Bernoulli admissible with a constant $C(m)$, provided that for every Banach space X, every finite set of functionals $x_1^*, \ldots, x_n^* \in X^*$, every family $(\varepsilon_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ of independent random Bernoulli variables on a nonatomic probability measure space $(\Omega, \mathcal{A}, \mathbb{P})$ and every sequence $(c_j)_{\mathbf{j}\in\mathcal{J}(m,n)}$ of complex numbers, we have

$$\int_\Omega \sup_{z\in B_X} \Big| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \varepsilon_{\mathbf{j}}(\omega)\, c_{\mathbf{j}}\, x_{\mathbf{j}}^*(z)\Big| \,\mathrm{d}\mathbb{P}(\omega)$$

$$\leqslant C(m)\big\|(c_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}\big\|_F \sup_{z\in B_X} \Big\| \sum_{k=1}^n x_k^*(z) e_k \Big\|_E^{m-1} J_\Phi(B_X, d),$$

where d is a pseudo-metric on B_X given by

$$d(z,w) = \Big\| \sum_{k=1}^n \big(x_k^*(z) - x_k^*(w)\big) e_k \Big\|_E, \quad (z,w) \in B_X \times B_X.$$

- Bayart (2012) For every $p \in [1,2]$, $E = \ell_p(\mathbb{N})$ and $\ell_\infty\big(|[j]|^{-1/p}\big)\big(\mathcal{J}(m,n)\big)$ are polynomially ψ_q-Bernoulli admissible, where $1/p + 1/q = 1$ and for $2 \leqslant q < \infty$ (i.e., $1 < p \leqslant 2$) the Orlicz function ψ_q is given by

$$\psi_q(t) = \exp(t^q) - 1, \quad t \geqslant 0,$$

and for $q = \infty$ (i.e., $p = 1$)

$$\psi_\infty(t) = \exp(\exp t) - e, \quad t \geqslant 0.$$

Lemma

Assume that X and $E^n = (\mathbb{C}^n, \|\cdot\|)$ are Banach spaces. For any set $\{x_1^*, \ldots, x_n^*\}$ of functionals in X^*, let d be a pseudo-metric on B_X given by

$$d(z,w) = \Big\| \sum_{k=1}^{n} \big(x_k^*(z) - x_k^*(w)\big) e_k \Big\|, \quad (z,w) \in B_X \times B_X.$$

Then, for every Orlicz function Φ we have

$$J_\Phi(B_X, d) \leqslant \Big(\sup_{z \in B_X} \sum_{k=1}^{n} |x_k^*(z)| \Big) J_\Phi(B_{\ell_1^n}, \|\cdot\|).$$

We show that, under some general hypotheses, certain Banach sequence spaces E and F modelled on $\mathbb{N}$ and $\mathcal{J}(m,n)$ are polynomially Φ-Bernoulli admissible.

- $\mathbb{C}^{\mathcal{M}(m,n),s}$ stands for all $(a_{\mathbf{i}}) \in \mathbb{C}^{\mathcal{M}(m,n)}$ for which $a_{\mathbf{i}} = a_{\mathbf{j}}$ whenever $\mathbf{j} \in [\mathbf{i}]$.
- For a Banach sequence space $F(\mathcal{M}(m,n))$ denote by $F^s(\mathcal{M}(m,n))$ the subspace $\mathbb{C}^{\mathcal{M}(m,n),s}$ of $F(\mathcal{M}(m,n))$. We define the mappings

$$\otimes_{m,n}\colon \mathbb{C}^{\mathcal{J}(m,n)} \times \mathbb{C}^{\mathcal{M}(m,n),s} \to \mathbb{C}^{\mathcal{J}(m,n)}, \quad \odot_{m,n}\colon (\mathbb{C}^n)^m \to \mathbb{C}^{\mathcal{M}(m,n)}$$

by the following formulas,

$$\otimes_{m,n}\big((a_{\mathbf{j}}),(b_{\mathbf{j}})\big) := (a_{\mathbf{j}} b_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$$

for all $(a_{\mathbf{j}}) \in \mathbb{C}^{\mathcal{J}(m,n)}$, $(b_{\mathbf{j}}) \in \mathbb{C}^{\mathcal{M}(m,n),s}$, and

$$\odot_{m,n}\big((x^{(1)}_{j_1})_{j_1=1}^n, \dots, (x^{(m)}_{j_m})_{j_m=1}^n\big) := \big(x^{(1)}_{j_1} \cdots x^{(m)}_{j_m}\big)_{(j_1,\dots,j_m)\in\mathcal{M}(m,n)}$$

for every $(x^{(k)}_{j_k})_{j_k=1}^n \in \mathbb{C}^n$.

- The set of all functions $\varphi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which are non-decreasing in each variable and positively homogeneous (that is, $\varphi(\lambda s, \lambda t) = \lambda\varphi(s,t)$ for all $\lambda, s, t \geqslant 0$) is denoted by $\mathcal{U}$. For any $\varphi \in \mathcal{U}$, we define $\varphi_*\colon (0,\infty) \times (0,\infty) \to (0,\infty)$ by

$$\varphi_*(s,t) = 1/\varphi(s^{-1}, t^{-1}), \quad s, t > 0.$$

Note that for any $\varphi \in \mathcal{U}$ the function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ given by $\rho(t) = \varphi(1,t)$ for all $t \geqslant 0$ is quasi-concave.

- The function φ which corresponds to an exact positive interpolation functor $\mathcal{F}$ is given by

$$\mathcal{F}(s\mathbb{R}, t\mathbb{R}) = \varphi(s,t)\mathbb{R}, \quad s, t > 0$$

is called the characteristic function of the functor $\mathcal{F}$. Here $\alpha\mathbb{R}$ denotes $\mathbb{R}$ equipped with the norm $\|\cdot\|_{\alpha\mathbb{R}} = \alpha|\cdot|$ for $\alpha > 0$.

Banach sequence spaces of Φ-Rademacher type

Definition Let Φ be an N-function and let L_Φ be the Orlicz space of functions on the nonatomic probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A Banach sequence space G modelled on a countable set I is said to be of Φ-Rademacher type if there is a constant $K > 0$ such that for every sequence (ε_i) of independent Bernoulli random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ and every $(x_i) \in G$ we have

$$\Big\| \sum_{i\in I} \varepsilon_i\, x_i \Big\|_{L_\Phi(\mathbb{P})} \leqslant K\, \|(x_i)\|_G.$$

Theorem

Suppose that $\mathcal{F}$ is an exact positive interpolation functor which is also a bounded lattice functor. Let ψ be the characteristic function of $\mathcal{F}$. Then the Banach sequence space

$$G(I) = \mathcal{F}(\ell_1(I), \ell_2(I))$$

modelled on any countable set I is of Φ-Rademacher type with $\Phi(t) = e^{\Psi(t)} - 1$ for all $t \geqslant 0$, where the Orlicz function Ψ satisfies $\Psi^{-1}(t) \asymp \psi_*(1, \sqrt{t})$.

Lemma

Given an Orlicz function Φ*, integers* $m \geqslant 2$*,* $n \geqslant 1$ *and Banach sequence lattices* $F(\mathcal{J}(m,n))$*,* $\widetilde{F}(\mathcal{M}(m,n))$ *and* $E(\mathbb{N})$*, if* $G(\mathcal{J}(m,n))$ *is of* Φ*-Rademacher type with a constant* $K > 0$ *and*

$$\|\otimes_{m,n} \colon F(\mathcal{J}(m,n)) \times \widetilde{F}^s(\mathcal{M}(m,n)) \to G(\mathcal{J}(m,n))\| \leqslant C_1(m),$$

$$\left\|\odot_{m,n} \colon (E^n)^m \to \widetilde{F}(\mathcal{M}(m,n))\right\| \leqslant C_2(m),$$

then $E(\mathbb{N})$ *and* $F(\mathcal{M}(m,n))$ *are polynomially* Φ*-Bernoulli admissible with a constant* $C(m) \leqslant mKC_1(m)C_2(m)$*.*

Theorem

Let $\mathcal{F}$ be an exact positive interpolation functor which is also a bounded lattice functor. Assume that the characteristic function of $\mathcal{F}$ satisfy satisfy $\psi(1,t) \to 0$ as $t \to 0+$ and $\psi(1,t) \to \infty$ as $t \to \infty$. If, for each $\mathcal{J} := \mathcal{J}(m,n)$ and $\mathcal{M} := \mathcal{M}(m,n)$ with $m \geqslant 2$ and $n \geqslant 1$, the following conditions are satisfied:

(i) *There exists a weight $w = (w_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}}$ such that*

$$\left\|\mathrm{id}\colon \mathcal{F}(\ell_1(w_0), \ell_2(w_1)) \to \mathcal{F}(\ell_1(\mathcal{J}), \ell_2(\mathcal{J}))(w)\right\| \leqslant C_1(m),$$

where the couple $(\ell_1(w_0), \ell_2(w_1))$ is modelled on $\mathcal{J}$ with $w_0 = (|[\mathbf{j}]|)$, $w_1 = \left(\sqrt{|[\mathbf{j}]|}\right)$.

(ii) $\|\odot_{m,n}\colon (E^n)^m \to \mathcal{F}(\ell_1(\mathcal{M}), \ell_2(\mathcal{M}))\| \leqslant C_2(m)$.

Then $E(\mathbb{N}) = \mathcal{F}(\ell_1(\mathbb{N}), \ell_2(\mathbb{N}))$ and $F(\mathcal{J}) = \ell_\infty(1/w)$ are polynomially Φ-Bernoulli admissible Banach spaces with the constant $C(m) \leqslant m\, C_1(m) C_2(m)$, where $\Phi(t) = e^{\Psi(t)} - 1$ for all $t \geqslant 0$ and Ψ is an Orlicz function such that $\Psi^{-1}(t) \asymp \psi_*(1, \sqrt{t})$.

Theorem

Assume that $\psi \in \widehat{\mathcal{U}}$ *is a super-multiplicative (i.e.,* $\psi(1,s)\psi(1,t) \leqslant \psi(1,st)$ *for all* $s,t>0$*) and satisfy* $\psi(1,t) \to 0$ *as* $t \to 0+$ *and* $\psi(1,t) \to \infty$ *as* $t \to \infty$*. Let* $\mathcal{J} = \mathcal{J}(m,n)$ *and* $\mathcal{M} = \mathcal{M}(m,n)$ *for each pair of integers* $m \geqslant 2$*,* $n \geqslant 1$ *and let* $w = (w_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}}$*, where* $w_{\mathbf{j}} = \rho\big(|[\mathbf{j}]|, \sqrt{|[\mathbf{j}]|}\big)$*, for each* $\mathbf{j} \in \mathcal{J}$*, and* ρ *is given by*

$$\rho(s,t) := \inf_{u,v>0} \frac{\psi(su,tv)}{\psi(u,v)}, \quad s,t \geqslant 0.$$

Then the Banach spaces $E(\mathbb{N}) = \psi(\ell_1(\mathbb{N}), \ell_2(\mathbb{N}))$ *and* $F(\mathcal{J}) = \ell_\infty(1/w)$ *are polynomially* Φ*-Bernoulli admissible with the constant* $C(m) \leqslant Km$*, where* K *is a universal constant and* $\Phi(t) = e^{\Psi(t)} - 1$ *for all* $t \geqslant 0$ *with* Ψ *an Orlicz function such that* $\Psi^{-1}(t) \asymp \psi(1,\sqrt{t})$*.*

An immediate consequence of the preceding theorem is the original result of Bayart (2012).

Corollary

Let $\mathcal{J} = \mathcal{J}(m, n)$ *and* $\mathcal{M} = \mathcal{M}(m, n)$ *for each pair of integers* $m \geqslant 2$, $n \geqslant 1$. *If* $1 < p \leqslant \infty$, *then* $\ell_p(\mathbb{N})$ *and* $F(\mathcal{J}) = \ell_\infty(|[\mathbf{j}]|^{-1/p})$ *are polynomially* Φ*-Bernoulli admissible Banach spaces with the constant* $C(m) \leqslant Km$ *for each* $m \in \mathbb{N}$, *where* K *is an absolute constant, and the Orlicz function* $\Phi(t) = e^{t^q} - 1$ *for all* $t \geqslant 0$, *where* $1/p + 1/q = 1$.

Theorem

Let $\mathcal{J} = \mathcal{J}(m, n)$ for each pair of integers $m \geqslant 2$, $n \geqslant 1$. Suppose that $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a concave and super-multiplicative function (i.e., $\theta(s)\theta(t) \leqslant \theta(st)$ for all s, $t > 0$). Let φ be an Orlicz function given by $\varphi^{-1}(t) = t\theta(1/\sqrt{t})$ for all $t > 0$. Then the Banach sequence spaces ℓ_φ and $F(\mathcal{J}) = \ell_\infty(1/\varphi^{-1}(|[\mathbf{j}]|)$ are polynomially Φ-Bernoulli admissible with the constant $C(m) \leqslant Km$ for some $K > 0$, where $\Phi(t) = e^{\Psi(t)} - 1$ with $\Psi^{-1}(t) = \theta(\sqrt{t})$ for all $t \geqslant 0$.

Estimates of entropy integrals (R. Szwedek & M. M)

- We recall that given a pseudo-metric (T, d), we denote by $N(T, d; \varepsilon)$ the entropy function associated with the pseudo-metric d on the set T for $\varepsilon > 0$, i.e., $N(T, d; \varepsilon)$ is the smallest number of open balls of radius $\varepsilon > 0$ in the pseudo-metric d needed to cover the set T.

- Given a Young function on $\mathbb{R}_+ := [0, \infty)$, i.e., $\Phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex increasing function such that $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$, the entropy integral of (T, d) with respect to Φ is defined by

$$J_\Phi(T, d) = \int_0^{\Delta(T)} \Phi^{-1}\big(N(T, d; \varepsilon)\big)\, d\varepsilon,$$

where $\Delta(T) = \sup_{s,t\in T} d(s, t)$ denotes the diameter of T.

Lemma

Assume that X and $E^n = (\mathbb{C}^n, \|\cdot\|)$ are Banach spaces. For any set $\{x_1^*, \dots, x_n^*\}$ of functionals in X^*, let d be a pseudo-metric on B_X given by

$$d(z,w) = \Big\| \sum_{k=1}^{n} \big(x_k^*(z) - x_k^*(w)\big) e_k \Big\|, \quad (z,w) \in B_X \times B_X.$$

Then, for every Orlicz function Φ we have

$$J_\Phi(B_X, d) \leqslant \Big(\sup_{z \in B_X} \sum_{k=1}^{n} |x_k^*(z)| \Big) J_\Phi(B_{\ell_1^n}, \|\cdot\|).$$

Theorem

Given a Young function Φ and positive integers $m \geqslant 2$ and $n \geqslant 1$, if Banach sequence spaces $E = E(\mathbb{N})$ and $F = F(\mathcal{J}(m,n))$ are polynomially Φ-Bernoulli admissible with the constant $C(m)$, then for every Banach space X, every finite set $\{x_1^*, \ldots, x_n^*\}$ of functionals in X^*, every family $(\varepsilon_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ of independent random Bernoulli variables on a nonatomic probability measure space $(\Omega, \mathcal{A}, \mathbb{P})$ and every sequence $(c_j)_{\mathbf{j}\in\mathcal{J}(m,n)}$ of complex numbers, we have

$$\int_\Omega \sup_{z\in B_X} \Big| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \varepsilon_{\mathbf{j}}(\omega) c_{\mathbf{j}} x_{\mathbf{j}}^*(z) \Big| \,\mathrm{d}\mathbb{P}(\omega)$$

$$\leqslant C(m) \big\| (c_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)} \big\|_F \sup_{z\in B_X} \Big\| \sum_{k=1}^n x_k^*(z) e_k \Big\|_E^{m-1} \Big(\sup_{z\in B_X} \sum_{k=1}^n |x_k^*(z)| \Big) J_\Phi(B_{\ell_1^n}, \|\cdot\|_{E^n}).$$

Corollary

Suppose that $\theta\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a concave and super-multiplicative function. Let* φ *be an Orlicz function given by* $\varphi^{-1}(t) = t\theta(1/\sqrt{t})$ *for all* $t > 0$*. Then there exists a constant* $K > 0$ *such that for every Banach space* X*, every finite set* $\{x_1^*, \ldots, x_n^*\}$ *of functionals in* X^**, every family* $(\varepsilon_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ *of independent random Bernoulli variables on a nonatomic probability measure space* $(\Omega, \mathcal{A}, \mathbb{P})$ *and every sequence* $(c_j)_{\mathbf{j}\in\mathcal{J}(m,n)}$ *of complex numbers, we have*

$$\int_\Omega \sup_{z\in B_X} \Big| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \varepsilon_{\mathbf{j}}(\omega) c_{\mathbf{j}} x_{\mathbf{j}}^*(z)\Big| \, \mathrm{d}\mathbb{P}(\omega)$$

$$\leqslant Km \sup_{\mathbf{j}\in\mathcal{J}(m,n)} \frac{|c_{\mathbf{j}}|}{\varphi^{-1}(|[\mathbf{j}]|)} \sup_{z\in B_X} \Big\| \sum_{k=1}^n x_k^*(z) e_k \Big\|_{\ell_\varphi}^{m-1} \Big(\sup_{z\in B_X} \sum_{k=1}^n |x_k^*(z)| \Big) J_\Phi(B_{\ell_1^n}, \|\cdot\|_{\ell_\varphi^n}),$$

where $\Phi(t) = e^{\Psi(t)} - 1$ *with* $\Psi^{-1}(t) = \theta(\sqrt{t})$ *for all* $t \geqslant 0$*.*

Definition The characteristic function of a Banach space A with respect to (A_0, A_1) is defined by

$$\psi_A(s,t) = \sup\left\{\|a\|_A;\ a \in A_0 \cap A_1,\ \|a\|_{A_0} \leqslant s,\ \|a\|_{A_1} \leqslant t\right\}, \quad s, t > 0.$$

Theorem

Let ϕ be a nondegenerate quasi-concave function ($\phi(0+) = 0$ and $\phi(t) \to \infty$ as $t \to \infty$) and let Φ be an Orlicz function such that $\Phi^{-1}(t) \asymp \phi(\log(1+t))$ for all $t \geqslant 0$. Let E be a complex Banach sequence space intermediate between ℓ_1 and ℓ_∞ and let $\psi = \psi_E(1,\cdot)$, where ψ_E is the characteristic function of E with respect to (ℓ_1, ℓ_∞). Then there exists a constant $C = C(\varphi, \psi) > 0$ such that

$$J_\Phi(B_{\ell_1^n}, \|\cdot\|_{E^n}) \leqslant C\Big(\phi(\log n) + \phi(n)\psi(1/n) + \int_{[\log n]}^{2n-1} \phi(s)\,\psi\Big(\frac{1}{s}\log\Big(1+\frac{n}{s}\Big)\Big)\frac{ds}{s}\Big), \quad n \geqslant 2.$$

Corollary

Assume that ϕ, $\psi_E(1,\cdot)$ and an Orlicz function Φ and the space E are defined as in the above Theorem. Then there exists a constant $C = C(\varphi,\psi) > 0$ such that the following statements about the entropy integral $J_\Phi(B_{\ell_1^n}, \|\cdot\|_{E^n})$ are true for each $n \geqslant 2$:

(i) *If the function $f\colon [1,\infty) \to (0,\infty)$ given by $f(s) = \phi(s)\,\psi\big(\frac{1}{s}\log\big(1+\frac{n}{s}\big)\big)$ for all $s \in [1,\infty)$ is almost decreasing for each integer $n \geqslant 2$ (that is, there exists $K > 0$ such that $f(s) \geqslant Kf(t)$ for every $1 \leqslant s < t < \infty$ and each $n \geqslant 2$), then*

$$J_\Phi(B_{\ell_1^n}, \|\cdot\|_{E^n}) \leqslant C\big(\phi(n)\psi(1/n) + \phi(\log n)\log n\big).$$

(ii) *If $\sup_{t\geqslant 1}\phi(t)\psi_E(1,1/t) < \infty$, then*

$$J_\Phi(B_{\ell_1^n}, \|\cdot\|_{E^n}) \leqslant C\,\bar{\phi}(\log n)\log n\,,$$

where in what follows $\bar{\phi}(t) := \sup_{s>0}\frac{\phi(st)}{\phi(s)}$ for all $t \geqslant 0$.

Corollary

Let φ be an N-function and let Φ be a Young function given by

$$\Phi^{-1}(t) \asymp \phi(\log(1+t)),$$

where $\phi(t) = t\varphi^{-1}(1/t)$ *for all* $t > 0$. *Then then there exists a constant* $C = C(\varphi) > 0$ *such that for each* $n \geqslant 2$ *we have*

$$J_\Phi(\ell_1^n, \|\cdot\|_{\ell_\varphi^n}) \leqslant C\,\bar{\phi}(\log n)\log n\,.$$

Applications

- We denote by $\mathcal{P}(^mX)$ the Banach space of all m-homogenous scalar-valued polynomials P on the Banach space X, equipped with the norm

$$\|P\|_{\mathcal{P}(^mX)} = \sup\big\{|P(x)|;\ \|x\| \leqslant 1\big\}.$$

  Given an n-dimension Banach space $X = (\mathbb{C}^n, \|\cdot\|)$, we exhibit some lower estimates of the unconditional basis constant for monomials z^α, $\alpha \in \mathbb{N}_0^n$, denoted by $\chi_{\text{mon}}(\mathcal{P}(^mX))$.

- Recall that a basis (x_n) in a Banach space X is said to be unconditional if there exists $C \geqslant 1$ such that

$$\Big\|\sum_{k=1}^{n} \theta_k \lambda_k x_k\Big\| \leqslant C \Big\|\sum_{k=1}^{n} \lambda_k x_k\Big\|$$

  for every choice of $(\lambda_k)_{k=1}^n$ and $(\theta_k)_{k=1}^n$ in $\mathbb{C}$ with $|\theta_k| = 1$; the smallest of the constants C is the unconditional basis constant of (x_n).

Theorem

Let $X = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space, and let φ be an Orlicz function given by $\varphi^{-1}(t) = t\theta(1/\sqrt{t})$ for all $t > 0$, where θ is a concave and super-multiplicative function. Then, for each $m \geqslant 2$ and $n \geqslant 2$, the following inequality holds true:

$$\frac{C\theta\big((m!)^{-1/2}\big)}{m\,\bar{\theta}\big(\sqrt{\log n}\big)\log n}\left(\frac{\sup_{z\in B_X}\sum_{k=1}^{n}|z_k|}{\sup_{z\in B_X}\|\sum_{k=1}^{n} z_k e_k\|_{\ell_\varphi}}\right)^{m-1} \leqslant \chi_{\mathrm{mon}}\big(\mathcal{P}(^m X)\big),$$

where $\bar{\theta}(t) = \sup_{s>0}\frac{\theta(st)}{\theta(s)}$ for all $t > 0$.

Remark:

In the case of ℓ_p-spaces with $p \in (1, 2]$, we recover the original result due to Bayart (2012).

Theorem

Let $X = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space, let $p \in (1, 2]$ and let q be its conjugate exponent. Then, for each $m \geqslant 2$ and $n \geqslant 2$, the following inequality holds true:

$$\frac{C}{m\,(m!)^{1/q}\,(\log n)^{1+1/q}}\left(\frac{\sup_{z\in B_X}\sum_{k=1}^{n}|z_k|}{\sup_{z\in B_X}\|\sum_{k=1}^{n} z_k e_k\|_{\ell_p}}\right)^{m-1} \leqslant \chi_{\mathrm{mon}}\big(\mathcal{P}(^m X)\big).$$

Corollary

Let φ be an Orlicz function given by $\varphi^{-1}(t) = t\theta(1/\sqrt{t})$ for all $t > 0$, where θ is a concave and super-multiplicative function. Then, for each $m \geqslant 2$ and $n \geqslant n$, the following inequality holds true:

$$\frac{C\theta\big((m!)^{-1/2}\big)\big(\theta(\sqrt{n})\big)^{m-1}}{m\,\bar{\theta}\big(\sqrt{\log n}\big)\log n} \leqslant \chi_{\mathrm{mon}}\big(\mathcal{P}({}^m\ell_\varphi^n)\big).$$