Universidad de Murcia | Departamento Matemáticas

Vector-valued Dirichlet series

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Some contents have been developped jointly with Prof. Andreas Defant.

1. Introduction

2. Bohr's transform
   - Scalar-valued
   - Vector-valued
   - Analytic Radon-Nikodym Property

Definition

A Dirichlet series is a formal expression

$$\sum_n a_n \frac{1}{n^s}$$

$$\sigma_a(D) = \inf\{\sigma \in \mathbb{R} : \sum_n a_n \frac{1}{n^s} \text{ converges absolutely on } [Re > \sigma]\}$$

$$\sigma_u(D) = \inf\{\sigma \in \mathbb{R} : \sum_n a_n \frac{1}{n^s} \text{ converges uniformly on } [Re > \sigma]\}$$

$$\sigma_c(D) = \inf\{\sigma \in \mathbb{R} : \sum_n a_n \frac{1}{n^s} \text{ converges on } [Re > \sigma]\}$$

Bohr's strip problem

$$T(X) := \sup\{\sigma_a(D) - \sigma_u(D) : \ D \text{ is } X\text{-valued Dirichlet series }\} =?$$

Theorem (A. Defant, D. García, M. Maestre, D. Pérez-García; 2008)

$$T(X) = 1 - \frac{1}{Cot(X)}$$

Recall that a Banach space X has **cotype** $p \in [2, +\infty)$ if there is a constant $C > 0$ such that for each choice of finitely many vectors $x_1, ..., x_n \in X$ we have

$$\left(\sum_{n=1}^{N} \|x_n\|^p \right)^{1/p} \leq C \left(\int_0^1 \left\| \sum_{n=1}^{N} r_n(t) x_n \right\|^2 dt \right)^{1/2}$$

The "best" cotype

$$Cot(X) := \inf \{2 \leq p \leq \infty : \ X \text{ has cotype } p\}$$

Bohr's transform

$H_\infty(\mathbb{T}^\mathbb{N}, X)$

$f : \mathbb{T}^\mathbb{N} \to X$
bounded measurable with

$$\hat{f}(\alpha) := \int_{\mathbb{T}^\mathbb{N}} f(\omega)\omega^{-\alpha}\, d\sigma$$

$$= 0 \text{ if } \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})}$$

$H_\infty(B_{c_0}, X)$

$F : B_{c_0} \to X$
bounded holomorphic

Monomial expansion:

$$F \sim \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha(F) z^\alpha$$

$\mathcal{H}_\infty(X)$

$D : [Re > 0] \to X$
bounded holomorphic

$$D(s) = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$$

whenever $s \in [Re > 0]$

Bohr's transform

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Scalar-valued Bohr transform: surjective isometries

$$H_\infty(\mathbb{T}^\mathbb{N}) \equiv H_\infty(B_{c_0}) \equiv \mathcal{H}_\infty$$

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \hat{f}(\alpha)\omega^\alpha \rightsquigarrow \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha(F) z^\alpha \rightsquigarrow \sum_{n \in \mathbb{N}} a_n \frac{1}{n^s}$$

where $\hat{f}(\alpha) = c_\alpha(F) = a_{p^\alpha}$ for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ $(p^\alpha := p_1^{\alpha_1} \dots p_k^{\alpha_k})$.

Bohr's transform

Vector valued surjective isometries????

$$H_\infty(\mathbb{T}^{\mathbb{N}}, X) \qquad H_\infty(B_{c_0}, X) \qquad \mathcal{H}_\infty(X)$$

$$\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} \hat{f}(\alpha)\omega^\alpha \quad \rightsquigarrow \quad \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} c_\alpha(F) z^\alpha \quad \rightsquigarrow \quad \sum_{n\in\mathbb{N}} a_n \frac{1}{n^s}$$

where $\hat{f}(\alpha) = c_\alpha(F) = a_{p^\alpha}$ for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ $\quad (p^\alpha := p_1^{\alpha_1} \ldots p_k^{\alpha_k})$.

The counterexample

$D \sim \sum_n e_n \frac{1}{(2^n)^s} \in \mathcal{H}_\infty(c_0)$

Bohr's transform

What works?

$$H_\infty(\mathbb{T}^{\mathbb{N}}, X) \quad \hookrightarrow \quad H_\infty(B_{c_0}, X) \quad \equiv \quad \mathcal{H}_\infty(X)$$

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \hat{f}(\alpha)\omega^\alpha \quad \rightsquigarrow \quad \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha(F) z^\alpha \quad \rightsquigarrow \quad \sum_{n \in \mathbb{N}} a_n \frac{1}{n^s}$$

where $\hat{f}(\alpha) = c_\alpha(F) = a_{p^\alpha}$ for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ $\quad (p^\alpha := p_1^{\alpha_1} \dots p_k^{\alpha_k})$.

Bohr's transform

Definition

$\mathcal{L}^+(L^1(\mathbb{T}^{\mathbb{N}}), X)$ *is the Banach space of operators* $T : L^1(\mathbb{T}^{\mathbb{N}}) \to X$ *with* $T(\omega^{-\alpha}) = 0$ *for every* $\alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})}$.

Vector-valued Bohr's transform

$$\mathcal{L}^+(L^1(\mathbb{T}^{\mathbb{N}}), X) \equiv H_\infty(B_{c_0}, X) \equiv \mathcal{H}_\infty(X)$$

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} T(\omega^{-\alpha})\omega^\alpha \rightsquigarrow \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha(F) z^\alpha \rightsquigarrow \sum_{n \in \mathbb{N}} a_n \frac{1}{n^s}$$

where $T(\omega^{-\alpha}) = c_\alpha(F) = a_{p^\alpha}$ for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ $\quad (p^\alpha := p_1^{\alpha_1} \dots p_k^{\alpha_k})$.

Fatou theorem

Theorem (*P. Fatou, Series trigonometriques et series de Taylor, Acta Math.* **30** *(1906), 335-400*)

Let f be a bounded holomorphic function on the open unit disc $\mathbb{D}$. Then, for almost every $w \in \mathbb{T}$ the limit

$$\tilde{f}(w) := \lim_{R \nearrow 1} f(Rw) \text{ exists}$$

Question

What about a function $f \in H_\infty(\mathbb{D}, E)$ for an arbitrary complex Banach space E?

ARNP

Definition (Bukhvalov-Danilevich)

E has the **Analytic Radon-Nikodym Property** *if given a function $f \in H_\infty(\mathbb{D}, E)$, for almost every $w \in \mathbb{T}$ the limit*

$$\tilde{f}(w) := \lim_{R \nearrow 1} f(Rw) \textit{ exists}$$

- RNP $\to$ ARNP
- $L^1(\mu)$ has ARNP
- c_0 does not have the ARNP

Bohr's transform

Theorem

X has the ARNP if and only if $H_\infty(\mathbb{T}^{\mathbb{N}}, X) \equiv H_\infty(B_{c_0}, X)$.

Lemma (Ingredient 1)

If X has ARNP and $G : \{Re > 0\} \to X$ is holomorphic and bounded then

$$\lim_{\sigma \searrow 0^+} G(\sigma + it) \text{ exists for almost every } t \in \mathbb{R}$$

Theorem (Ingredient 2: Ergodic Theorem)

Suppose that $(T_t)_{t\geq 0}$ is an ergodic (measure preserving) semiflow on (Ω, Σ, μ). If $g \in L^p(\mu)$ then

$$\lim_{R\to+\infty} \frac{1}{R} \int_0^R g(T_t\omega)\, dt = \int_\Omega g\, d\mu \qquad \text{for almost every } \omega \in \Omega.$$

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Boundary properties of analytic and harmonic functions with values in a Banach space, Mat. Zametki, vol. 31, 1982.

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Integral means and boundary limits of Dirichlet series, Bull. Lond. Math. Soc., vol 41, num 3, 2009.

K. Petersen,
Ergodic Theory, Cambridge University Press, 1989.

Post-credit frame

Let C be a bounded convex closed subset of a Banach space E.

James theorem

If every $x^* \in E^*$ attains its supremum on C then C is weakly compact.

One-side James theorem

Suppose that C is weakly K-determined. If every $x^* \in E^*$ with

$$\sup \{\langle x^*, x\rangle : x \in C\} < 0$$

attains its supremum on C then C is weakly compact.

Examples: separable spaces, $L^1(\mu)$-spaces

Coming soon...