Entropy, spectra and width numbers of operators

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Valencia 2017

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Spectral radius formula

- The spectrum $\sigma(T)$ of T on a complex Banach space X

$$\sigma(T) := \left\{ \lambda \in \mathbb{C} \ ; \ \lambda I_X - T \ \text{ is not invertible in } \ L(X) \right\}$$

- $\sigma(T)$ is contained in the circle of radius

$$\lim_{m\to\infty} \|T^m\|^{1/m}$$

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- $\sigma(T)$ is not empty and $\rho(T) = \mathbb{C} \backslash \sigma(T)$ is open
- Gelfand's spectral radius formula

$$r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{m \to \infty} \|T^m\|^{1/m}$$

Eigenvalues of a compact operator

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Eigenvalues of a compact operator

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- $\dim N(\lambda I - T)$ is called the geometric multiplicity of $\lambda \neq 0$
- $\dim N_\infty(\lambda I_X - T)$, where

$$N_\infty(\lambda I_X - T) := \bigcup_{n=1}^{\infty} N\big((\lambda I_X - T)^n\big)$$

is called the algebraic multiplicity of $\lambda \neq 0$

Definition

$S \in L(X, Y)$ is called a Fredholm operator if

$$\dim N(S) < \infty \quad \text{and} \quad \operatorname{codim} R(S) < \infty$$

- The essential spectrum $\sigma_{ess}(T)$ of T

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- $S \in L(X)$ is a Fredholm operator $\Longleftrightarrow$ its equivalence class $\overline{S}$ is invertible in the Calkin algebra $L(X)/K(X)$.

$$\sigma_{ess}(T) = \sigma(\overline{T})$$

The essential spectral radius

- The essential spectral radius

$$r_{ess}(T) := \sup_{\lambda \in \sigma_{ess}(T)} |\lambda| = \lim_{m \to \infty} \|T^m\|_{ess}^{1/m}$$

- The Riesz part of the spectrum

$$\Lambda(T) := \{ \lambda \in \sigma(T) \ : \ |\lambda| > r_{ess}(T) \}$$

is at most countable and consists of isolated eigenvalues of finite algebraic multiplicity.

Eigenvalue sequence for an arbitrary operator (1)

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Eigenvalue sequence for an arbitrary operator (1)

- The Riesz part of the spectrum

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We assign an eigenvalue sequence $\{\lambda_n(T)\}_{n=1}^{\infty}$ for $T \in L(X)$ from the elements of the set $\Lambda(T) \cup \{r_{ess}(T)\}$ as follows:

- The eigenvalues are arranged in an order of non-increasing absolute values.
- Every eigenvalue $\lambda \in \Lambda(T)$ is counted according to its algebraic multiplicity.
- If T possesses less than n eigenvalues λ with $|\lambda| > r_{ess}(T)$, we let

$$\lambda_n(T) = \lambda_{n+1}(T) = \ldots = r_{ess}(T)$$

Eigenvalue sequence for an arbitrary operator (2)

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Remark

$$r(T) = |\lambda_1(T)| \quad \text{and} \quad r_{ess}(T) = \lim_{n\to\infty} |\lambda_n(T)|$$

Entropy numbers

Definition

The n-th entropy number $\varepsilon_n(T)$ of $T \in L(X, Y)$ is defined by

$$\varepsilon_n(T) := \inf\left\{ \varepsilon > 0 \ : \ T(U_X) \subset \bigcup_{i=1}^{n} \{y_i + \varepsilon U_Y\}, \quad y_i \in Y \right\}$$

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- Entropy numbers are: monotonne

$$0 \leqslant \ldots \leqslant \varepsilon_3(T) \leqslant \varepsilon_2(T) \leqslant \varepsilon_1(T) = \|T\|$$

- sub-multiplicative

$$\varepsilon_{kl}(RS) \leqslant \varepsilon_k(R)\,\varepsilon_l(S) \quad \text{for} \quad S \in L(X, Z) \quad \text{and} \quad R \in L(Z, Y)$$

- sub-additive

$$\varepsilon_{kl}(T_1 + T_2) \leqslant \varepsilon_k(T_1) + \varepsilon_l(T_2) \quad \text{for} \quad T_1, T_2 \in L(X, Y).$$

Measure of non-compactness

- The measure of non-compactness

$$\beta(T) := \lim_{n\to\infty} \varepsilon_n(T)$$

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$$\beta(T) := \lim_{n\to\infty} \varepsilon_n(T)$$

$$\beta(T) \leqslant \|T\|_{ess}$$

- Nussbaum's formula [1970]. Let X be a complex Banach space and $T \in L(X)$, then

$$r_{ess}(T) = \lim_{m\to\infty} \beta(T^m)^{1/m}$$

Carl-Triebel's inequality [1980]

Let $\{\lambda_n(T)\}_{n=1}^{\infty}$ be an eigenvalue sequence of $T \in K(X)$ on a complex Banach space X.

- Carl's inequality

$$|\lambda_n(T)| \leqslant \sqrt{2}\, e_n(T) \quad \text{where} \quad e_n(T) := \varepsilon_{2^{n-1}}(T)$$

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$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} \leqslant \inf_{k\in\mathbb{N}} k^{1/(2n)} \varepsilon_k(T)$$

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- Makai and Zemánek proved the inequality for arbitrary $T \in L(X)$ [1980]

Banach couple

- We call $\vec{A} := (A_0, A_1)$ a Banach couple if both A_0 and A_1 are Banach spaces such that

$$A_0, A_1 \hookrightarrow \mathcal{X}$$

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For a given Banach couple $\vec{A}$, we define spaces

- intersection $A_0 \cap A_1$ with the norm

$$\|a\|_{A_0 \cap A_1} = \max\left\{ \|a\|_{A_0}, \|a\|_{A_1} \right\}$$

- sum $A_0 + A_1$ with the norm

$$\|a\|_{A_0 + A_1} = \inf_{a = a_0 + a_1} \left\{ \|a_0\|_{A_0} + \|a_1\|_{A_1} \right\}$$

Interpolation functor

- By $T\colon \vec{A} \to \vec{B}$ we denote an operator $T\colon A_0 + A_1 \to B_0 + B_1$, such that

$$T|_{A_j} \in L(A_j, B_j), \quad j = 0, 1$$

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Definition

By an interpolation functor we mean a mapping $\mathcal{F}\colon \vec{\mathcal{B}} \to \mathcal{B}$

- $A_0 \cap A_1 \subset \mathcal{F}(\vec{A}) \subset A_0 + A_1$ for any $\vec{A} \in \vec{\mathcal{B}}$
- $T\big(\mathcal{F}(\vec{A})\big) \subset \mathcal{F}(\vec{B})$ for any $\vec{A}, \vec{B} \in \vec{\mathcal{B}}$ and $T\colon \vec{A} \to \vec{B}$

Interpolation functor of exponential type of θ

For all interpolation functors $\mathcal{F}$

$$\| T \|_{\mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})} \leqslant C \max\left\{ \| T \|_{A_0 \to B_0}, \| T \|_{A_1 \to B_1} \right\}$$

Interpolation functor of exponential type of θ

For all interpolation functors $\mathcal{F}$

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \leqslant C\,\max\left\{\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}\right\}$$

If in addition there exists $\theta \in (0,1)$ such that

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \leqslant C\,\|T\|_{A_0\to B_0}^{1-\theta}\,\|T\|_{A_1\to B_1}^{\theta},$$

then $\mathcal{F}$ is called of exponential type of θ.

- The real $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$ and complex $\mathcal{F}(\cdot) = [\cdot]_\theta$ interpolation functors are of exponential type of θ.

Recollect...

- The n-th entropy number $\varepsilon_n(T)$ of $T \in L(X, Y)$

$$\varepsilon_n(T) := \inf\left\{ \varepsilon > 0 \ : \ T(U_X) \subset \bigcup_{i=1}^{n} \{y_i + \varepsilon U_Y\}, \quad y_i \in Y \right\}$$

- The measure of non-compactness

$$\beta(T) := \lim_{n\to\infty} \varepsilon_n(T)$$

A delicate problem

Let $\mathcal{F}$ be an interpolation functor of exponential type of θ. Does there exist a constant $C > 0$ such that for any $T\colon \vec{A} \to \vec{B}$

$$\beta\big(T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})\big) \leqslant C\,\beta\big(T\colon A_0 \to B_0\big)^{1-\theta}\beta\big(T\colon A_1 \to B_1\big)^{\theta} \text{ ?}$$

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This question was answered positively

- for the real interpolation functor $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$ by [Cobos, Fernández-Martínez and Martínez 1999], [Fernández-Martínez 2006], [Szwedek 2006],
- for the complex interpolation functor $\mathcal{F}(\cdot) = [\cdot]_\theta$ in the case where $\vec{B}$ satisfies an approximation condition by [Teixeira and Edmunds 1981], [Szwedek 2015].

A more delicate problem

Let $\mathcal{F}$ be an interpolation functor of exponential type of θ. Does there exist a constant $C > 0$ such that for any $T\colon \vec{A} \to \vec{B}$

$$\varepsilon_{k_0 k_1}\big(T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})\big) \leqslant C\, \varepsilon_{k_0}(T\colon A_0 \to B_0)^{1-\theta}\, \varepsilon_{k_1}(T\colon A_1 \to B_1)^{\theta} \text{ ?}$$

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The reduction [Mastyło, Szwedek 2017]

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Theorem [Mastyło, Szwedek 2017]

Suppose that $\mathcal{F}$ is an interpolation functor of exponential type of θ. If $T : \vec{A} \to \vec{A}$, then

$$\left|\lambda_n\left(T|_{\mathcal{F}(\vec{A})}\right)\right| \leqslant 2e_n(T|_{A_0})^{1-\theta}\, e_n(T|_{A_1})^{\theta}$$

and

$$\left(\prod_{i=1}^{n}\left|\lambda_i\left(T|_{\mathcal{F}(\vec{A})}\right)\right|\right)^{1/n} \leqslant \inf_{k_0,k_1\in\mathbb{N}} (k_0 k_1)^{1/2n}\, \varepsilon_{k_0}(T|_{A_0})^{1-\theta}\, \varepsilon_{k_1}(T|_{A_1})^{\theta}$$

Generalizations of the spectral radius formula (1)

- Gelfand's spectral radius formula

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Definition

Given $T \in L(X)$, the n-th *approximation number* is defined by

$$a_n(T) := \inf\left\{ \|T - S\| \ : \ S \in L(E,F) \ \text{with} \ \operatorname{rank}(S) < n \right\}$$

- König's [1978] formula; a generalization for higher eigenvalues

$$|\lambda_n(T)| = \lim_{m\to\infty} a_n(T^m)^{1/m}$$

Generalizations of the spectral radius formula (2)

$$\left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n} \leqslant \inf_{k\in\mathbb{N}} k^{1/(2n)} \varepsilon_k(T)$$

- The n-th entropy modulus $g_n(T)$ of $T \in L(X)$ is given by

$$\left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} \leqslant \inf_{k \in \mathbb{N}} k^{1/(2n)} \varepsilon_k(T) =: g_n(T)$$

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- Makai-Zemánek's formula [1982]

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<u>Problem</u>

In what form does it exist a formula for the spectral radius of T using the entropy numbers of powers of operators?

Spectral entropy numbers (1)

Theorem [Mastyło, Szwedek 2017]

Let X be a complex Banach space and $T \in L(X)$. If $\{\lambda_n(T)\}$ is an eigenvalue sequence of T, then

$$\sup_{n\in\mathbb{N}} k^{-1/(2n)} \left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} = \lim_{m\to\infty} \varepsilon_{k^m}(T^m)^{1/m}$$

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We define the k-th *spectral entropy number* $\mathcal{E}_k(T)$ by

$$\mathcal{E}_k(T) := \lim_{m\to\infty} \varepsilon_{k^m}(T^m)^{1/m}$$

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Spectral entropy numbers (2)

Theorem [Mastyło, Szwedek 2017]

Fix $t \in [1,\infty)$. If $\{t_m\} \subset \mathbb{N}$ is such that $\lim_{m\to\infty} t_m^{1/m} = t$, then

$$\sup_{n\in\mathbb{N}} t^{-1/(2n)} \left(\prod_{i=1}^{n} |\lambda_i(T)| \right)^{1/n} = \lim_{m\to\infty} \varepsilon_{t_m}(T^m)^{1/m}$$

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Definition

Define the *spectral entropy map* $t \mapsto \mathcal{E}_t(T)$ of T as follows

$$\mathcal{E}_t(T) := \lim_{m\to\infty} \varepsilon_{t_m}(T^m)^{1/m}$$

Spectral entropy numbers (3)

Proposition [Mastyło, Szwedek 2017]

Let X be a complex Banach space and $T \in L(X)$. If $\{\lambda_n(T)\}$ is an eigenvalue sequence of T, then

$$\lim_{m\to\infty} \varepsilon_n(T^m)^{1/m} = \mathcal{E}_1(T) = r(T) \;\; \text{for each} \;\; n \in \mathbb{N}$$

$$\lim_{m\to\infty} \varepsilon_m(T^m)^{1/m} = \mathcal{E}_1(T)$$

$$\lim_{m\to\infty} e_m(T^m)^{1/m} = \mathcal{E}_2(T) = \sup_{n\in\mathbb{N}} \left(\frac{\prod_{i=1}^{n} |\lambda_i(T)|}{\sqrt{2}} \right)^{1/n}$$

$$\mathcal{E}_\infty(T) := \lim_{t\to\infty} \mathcal{E}_t(T) = r_{ess}(T)$$

Spectral entropy numbers (4)

Proposition [Mastyło, Szwedek 2017]

Let X, Y be a complex Banach space.

- *If $R, S \in L(X)$ are commuting operators, then*

$$\mathcal{E}_{tu}(RS) \leqslant \mathcal{E}_t(R)\,\mathcal{E}_u(S)\,, \quad t, u \in [1, \infty].$$

- $$\mathcal{E}_{t^n}(R^n) = \mathcal{E}_t(R)^n\,, \quad t \in [1, \infty],\ n \in \mathbb{N}.$$

- *If $T \in L(X, Y)$ and $U \in L(Y, X)$, then*

$$\mathcal{E}_t(TU) = \mathcal{E}_t(UT)\,, \quad t \in [1, \infty].$$

Entropy moduli

Definition

Given an operator $T \in L(X)$ on a complex Banach space X, we define the entropy modulus $g_n \ (T)$ as follows

$$g_n \ (T) := \inf_{k\in\mathbb{N}} k^{1/(2n)} \quad \varepsilon_k(T) \ , \quad n \in \mathbb{N}$$

Entropy moduli

Definition

Given an operator $T \in L(X)$ on a complex Banach space X, we define the entropy modulus $g_s \ (T)$ as follows

$$g_s \ (T) := \inf_{k \in \mathbb{N}} k^{1/(2s)} \quad \varepsilon_k(T) \ , \quad s \in (0, \infty)$$

Entropy moduli

Definition

Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a sub-multiplicative function.
Given an operator $T \in L(X)$ on a complex Banach space X, we define the entropy modulus $g_{s,\varphi}(T)$ as follows

$$g_{s,\varphi}(T) := \inf_{k\in\mathbb{N}} k^{1/(2s)}\varphi\left(\varepsilon_k(T)\right), \qquad s \in (0,\infty)$$

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$$g_{s,\varphi}(T) := \inf_{k\in\mathbb{N}} k^{1/(2s)}\varphi\left(\varepsilon_k(T)\right), \quad s \in (0,\infty)$$

- Denote by $\widetilde{\varphi}$ the function on $[0,\infty)$ given by

$$\widetilde{\varphi}(u) := \lim_{m\to\infty} \varphi(u^m)^{1/m}, \quad u \geqslant 0$$

- $\widetilde{\varphi}$ is sub-multiplicative and $\widetilde{\varphi} \leqslant \varphi$

Theorem [Mastyło, Szwedek 2017]

Let X be an arbitrary complex Banach space and $T \in L(X)$. Assume that $\varphi\colon [0,\infty) \to [0,\infty)$ is a nondecreasing, sub-multiplicative and right-continuous function. Then

$$\inf_{t\in[1,\infty)} t^{1/(2s)}\widetilde{\varphi}(\mathcal{E}_t(T)) = \lim_{m\to\infty} g_{s,\varphi}(T^m)^{1/m}, \quad s \in (0,\infty)$$

Makai-Zemánek's formula revisited

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In particular,

$$\inf_{t\in[1,\infty)} t^{1/(2n)}\mathcal{E}_t(T) = \left(\prod_{i=1}^{n} |\lambda_i(T)|\right)^{1/n}$$

Interpolation of spectral entropy numbers holds

Theorem [Mastyło, Szwedek 2017]

If $\mathcal{F}$ be an interpolation functor of exponential type of θ, then for any $T\colon \vec{A} \to \vec{A}$

$$\mathcal{E}_{k_0 k_1}\big(T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{A})\big) \leqslant \mathcal{E}_{k_0}(T\colon A_0 \to A_0)^{1-\theta}\,\mathcal{E}_{k_1}(T\colon A_1 \to A_1)^{\theta}$$

Approximation numbers

Definition

The n-th approximation number $a_n(T)$ of $T \in L(E, F)$ is defined by

$$a_n(T) := \inf\Big\{ \|T - S\| \ : \ S \in L(E, F) \ \text{with} \ \text{rank}(S) < n \Big\}$$

Approximation numbers

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- Monotonicity:

$$\|T\| = a_1(T) \geqslant a_2(T) \geqslant a_3(T) \geqslant \ldots \geqslant 0$$

- Sub-additivity:

$$a_{k+n-1}(T_1 + T_2) \leqslant a_k(T_1) + a_n(T_2) \quad \text{for} \quad T_1, T_2 \in L(E,F)$$

- Sub-multiplicativity:

$$a_{k+n-1}(RS) \leqslant a_k(R)\, a_n(S) \quad \text{for} \quad S \in L(E,Z), R \in L(Z,F)$$

Kolmogorov and Gelfand numbers of $T \in L(E,F)$

- Approximation numbers:

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- Kolmogorov numbers:

$$d_n(T) := \inf\Big\{ \varepsilon > 0 \ : \ G \subset F, \dim(G) < n, \ T(U_E) \subset G + \varepsilon U_F \Big\}$$

- Gelfand numbers:

$$c_n(T) := \inf\Big\{ \|T|_G\| \ : \ G \subset E, \operatorname{codim}(G) < n \Big\}$$

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How to express the growth of $a_n(T)$ relative to $d_n(T)$ and $c_n(T)$?

$$d_n(T) \leqslant a_n(T) \leqslant (2n)^{1/2} d_n(T) \,, \ \ c_n(T) \leqslant a_n(T) \leqslant (2n)^{1/2} c_n(T)$$

- We call $\vec{A} := (A_0, A_1)$ a Banach couple if both A_0 and A_1 are Banach spaces such that

$$A_0, A_1 \hookrightarrow \mathcal{X}$$

For a given Banach couple $\vec{A}$, we define spaces

- intersection $A_0 \cap A_1$ with the norm

$$\|a\|_{A_0 \cap A_1} = \max\left\{\|a\|_{A_0}, \|a\|_{A_1}\right\}$$

- sum $A_0 + A_1$ with the norm

$$\|a\|_{A_0 + A_1} = \inf_{a = a_0 + a_1}\left\{\|a_0\|_{A_0} + \|a_1\|_{A_1}\right\}$$

- By $T\colon \vec{A} \to \vec{B}$ we denote an operator $T\colon A_0 + A_1 \to B_0 + B_1$, such that

$$T|_{A_0} \in L(A_0, B_0) \;\text{ and }\; T|_{A_1} \in L(A_1, B_1)$$

Definition

By an interpolation functor we mean a mapping $\mathcal{F}\colon \vec{\mathcal{B}} \to \mathcal{B}$

- $A_0 \cap A_1 \subset \mathcal{F}(\vec{A}) \subset A_0 + A_1$ for any $\vec{A} \in \vec{\mathcal{B}}$
- $T\big(\mathcal{F}(\vec{A})\big) \subset \mathcal{F}(\vec{B})$ for any $\vec{A}, \vec{B} \in \vec{\mathcal{B}}$ and $T\colon \vec{A} \to \vec{B}$

Recollect...Interpolation functor of exponential type of θ

For all interpolation functors $\mathcal{F}$

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \leqslant C \max\left\{\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}\right\}$$

If in addition there exists $\theta \in (0,1)$ such that

$$\|T\|_{\mathcal{F}(\vec{A})\to\mathcal{F}(\vec{B})} \leqslant C \, \|T\|_{A_0\to B_0}^{1-\theta} \, \|T\|_{A_1\to B_1}^{\theta},$$

then $\mathcal{F}$ is called of exponential type of θ.

- The real $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$ and complex $\mathcal{F}(\cdot) = [\cdot]_\theta$ interpolation functors are of exponential type of θ.

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One-sided interpolation results

Let $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ be Banach couples and $\theta \in (0,1)$.

- If X belongs to the class $\mathcal{C}_K(\theta; \vec{A})$ and $B := B_0 = B_1$, then

  $$d_{n+k-1}(T\colon X \to B) \leqslant C\, d_n(T\colon A_0 \to B)^{1-\theta}\, d_k(T\colon A_1 \to B)^{\theta}\,.$$

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- If $A := A_0 = A_1$ and Y belongs to the class $\mathcal{C}_J(\theta; \vec{B})$, then

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- The real $\mathcal{F}(\cdot) = (\cdot)_{\theta,q}$ and complex $\mathcal{F}(\cdot) = [\cdot]_\theta$ interpolation functors are members of $\mathcal{C}_K(\theta; \cdot)$ and $\mathcal{C}_J(\theta; \cdot)$.

A subtle problem

Let $\mathcal{F}$ be an interpolation functor of exponential type of θ. Does there exist a constant $C > 0$ such that for any $T\colon \vec{A} \to \vec{B}$

$$d_{n+k-1}\big(T\colon \mathcal{F}(\vec{A}) \to \mathcal{F}(\vec{B})\big) \leqslant C\, d_n(T\colon A_0 \to B_0)^{1-\theta}\, d_k(T\colon A_1 \to B_1)^{\theta}$$

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Two-sided interpolation of $d_n(T)$, $c_n(T)$ fails

The observation of Carl

Consider $A_0 = A_1 = B_0 := \ell_1^{3n}$, $B_1 := \ell_\infty^{3n}$ and $\theta = 1/2$.

Then $X := [A_0, A_1]_\theta \cong \ell_1^{3n}$ and $Y := [B_0, B_1]_\theta \cong \ell_2^{3n}$.

Therefore

- $$d_{2n-1}(I \colon X \to Y) \asymp 3^{-1/2}.$$
- $$d_n(I \colon A_0 \to B_0)^{1-\theta}\, d_n(I \colon A_1 \to B_1)^{\theta} \asymp n^{-1/4}.$$

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- $$d_n(I\colon A_0 \to B_0)^{1-\theta}\, d_n(I\colon A_1 \to B_1)^{\theta} \asymp n^{-1/4}.$$
- The duality relation $c_n(T) = d_n(T')$.

Interpolation of $a_n(T)$ between Hilbert spaces

Theorem [Szwedek 2015]

There exists a constant $C > 0$, such that

- *for all couples $\vec{H} = (H_0, H_1)$ and $\vec{K} = (K_0, K_1)$ of complex Hilbert spaces, and*
- *for every operator $T \in L(\vec{H}, \vec{K})$ and every $\theta \in (0,1)$*

$$A_n\big(T\colon [\vec{H}]_\theta \to [\vec{K}]_\theta\big) \leqslant C\, A_n\big(T\colon H_0 \to K_0\big)^{1-\theta} A_n\big(T\colon H_1 \to K_1\big)^{\theta},$$

where $A_n(T)$ denotes $\big(\prod_{i=1}^{n} a_i(T)\big)^{1/n}$.

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