Unconditional atomic decompositions in Fréchet spaces

XI Encuentro de Análisis Funcional Alicante - Murcia - València

Alicante

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Our aim is discuss unconditional atomic decompositions on no normable Fréchet Spaces.

Atomic Decompositions have been investigated by Casazza, Gröchenig, Carando, Lassalle, Schmidberg, Korobeĭnik, Taskinen and others.
Introduction

Atomic Decompositions

Duality

Unconditional atomic decompositions

Example
Definition

Let $E$ be a Hausdorff locally convex space.
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**Definition**

Let $\{x_j\}_{j=1}^\infty \subset E$ and let $\{x'_j\}_{j=1}^\infty \subset E'$, we say that $(\{x'_j\}, \{x_j\})$ is an atomic decomposition of $E$ if

$$x = \sum_{j=1}^\infty x'_j (x) x_j, \quad \text{for all } x \in E,$$

the series converging in $E$. 
Let $E$ be a Hausdorff locally convex space.

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Let $E$ be a Hausdorff locally convex space.

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the series converging in $E$.

We denote by $\omega$ the space $\mathbb{K}^\mathbb{N}$ endowed by the product topology. A sequence space is a lcs $\bigwedge$ such that $\mathbb{K}^{(\mathbb{N})} \subset \bigwedge \subset \omega$, this last inclusion being continuous.
Example (Leont’ev, 1970’s)

For every convex bounded set, $\Omega \subset \mathbb{C}$, there exists a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{C}$ such that, for every $f \in \mathcal{H}(\Omega)$,

$$f(z) = \sum_{j=1}^{\infty} c_j e^{x_j z}$$

is uniformly and absolutely convergent on compact sets.
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is uniformly and absolutely convergent on compact sets. The sequence \( \{c_j\}_{j=1}^{\infty} \) is not unique, therefore it is not a basis.

Korobeňnik, Y. F. and Melikhov, S. N. proved that, if the boundary of \( \Omega \) is \( C^2 \), there exist \( \{c_j\}_{j=1}^{\infty} \) depending continuously of \( f \) (i.e. \( c_j := u_j(f) \) where \( u_j \) is a linear and continuous operator). Therefore, we obtain an atomic decomposition.
**Example**

Let $E$ be a lcs with a Schauder basis $\{e_j\}_{j=1}^{\infty} \subset E$ and denote by $\{e'_j\}_{j=1}^{\infty} \subset E'$ the functional coefficients. Then $\left(\{e'_j\}, \{e_j\}\right)$ is an atomic decomposition for $E$ such that $e'_j(e_i) = \delta_{j,i}$ for all $i,j \in \mathbb{N}$. 
Example

Let $E$ be a lcs with a Schauder basis $\{e_j\}_{j=1}^{\infty} \subset E$ and denote by $\{e'_j\}_{j=1}^{\infty} \subset E'$ the functional coefficients. Then $(\{e'_j\}, \{e_j\})$ is an atomic decomposition for $E$ such that $e'_j(e_i) = \delta_{j,i}$ for all $i,j \in \mathbb{N}$.

Example

Let $E$ be a lcs and let $P : E \to E$ be a continuous linear projection. If $(\{x'_j\}, \{x_j\})$ is an atomic decomposition for $E$, then $(\{P'(x'_j)\}, \{P(x_j)\})$ is an atomic decomposition for $P(E)$.
Outline

1. Introduction

2. Atomic Decompositions

3. Duality

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5. Example
Theorem

From now on $E$ always be a barrelled and complete Hausdorff locally convex space.
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The following are equivalent:

1. $E$ admits an atomic decomposition.
2. $E$ is isomorphic to a complemented subspace of a complete sequence space with the canonical unit vectors as Schauder basis.

**Theorem**

A Fréchet space $E$ admits an atomic decomposition if, and only if, $E$ has the bounded approximation property.
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From now on $E$ always be a barrelled and complete Hausdorff locally convex space.

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1. $E$ admits an atomic decomposition.
2. $E$ is isomorphic to a complemented subspace of a complete sequence space with the canonical unit vectors as Schauder basis.
3. $E$ is isomorphic to a complemented subspace of a complete sequence space with Schauder basis.
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The following are equivalent:

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A Fréchet space \( E \) admits an atomic decomposition if, and only if, \( E \) has the bounded approximation property.
1) ⇒ 2)

- We define an injective and continuous linear map

\[ U : E \rightarrow \bigwedge \]

\[ x \mapsto U(x) := \left( x'_j(x) \right)_j . \]
Sketch of the Proof

1) $\Rightarrow$ 2)

- We define an injective and continuous linear map

$$U : E \rightarrow \Lambda$$

$$x \rightarrow U(x) := \left(x'_j(x)\right)_j.$$  

Where $\Lambda := \{ \alpha = (\alpha_j)_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j \text{ is convergent in } E \}$ is a sequence space endowed with the system of seminorms

$$Q := \left\{ q_p((\alpha_j)_j) := \sup_{n} p\left(\sum_{j=1}^{n} \alpha_j x_j\right), \text{ for all } p \in cs(E) \right\},$$

such that $(\Lambda, Q)$ is complete.
Sketch of the Proof

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such that $(\bigwedge, Q)$ is complete.

- And $S : \bigwedge \rightarrow E$, $S((\alpha_j)_j) := \sum_{j=1}^{\infty} \alpha_j x_j$, is a continuous linear map.
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- We define a injective and continuous linear linear map

\[ U : E \longrightarrow \bigwedge, \quad x \mapsto U(x) := (x'_j(x))_j. \]

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\[ Q := \left\{ q_p((\alpha_j)_j) := \sup_n p\left( \sum_{j=1}^{n} \alpha_j x_j \right), \text{ for all } p \in cs(E) \right\}, \]

such that \((\bigwedge, Q)\) is complete.

- And \( S : \bigwedge \longrightarrow E, \ S((\alpha_j)_j) := \sum_{j=1}^{\infty} \alpha_j x_j \), is a continuous linear map.

- From \( S \circ U = I_E \) we conclude that \( U \) is an isomorphism into its range \( U(E) \).
1) $\Rightarrow$ 2)

- We define a injective and continuous linear linear map

$$U : E \rightarrow \bigwedge$$

$$x \rightarrow U(x) := \left( x'_j(x) \right)_j.$$ 

Where $\bigwedge := \{ \alpha = (\alpha_j)_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j \text{ is convergent in } E \}$ is a sequence space endowed with the system of seminorms

$$Q := \left\{ q_p((\alpha_j)_j) := \sup_{n} p\left( \sum_{j=1}^{n} \alpha_j x_j \right), \text{ for all } p \in cs(E) \right\},$$

such that $(\bigwedge, Q)$ is complete.

- And $S : \bigwedge \rightarrow E$, $S((\alpha_j)_j) := \sum_{j=1}^{\infty} \alpha_j x_j$, is a continuous linear map.

- From $S \circ U = I_E$ we conclude that $U$ is an isomorphism into its range $U(E)$ and $U \circ S$ is a projection of $\bigwedge$ onto $U(E)$. 
Theorem

Let \( \{x'_j\}, \{x_j\} \) be an atomic decomposition of a complete lcs \( E \). Then, if \( \{y_j\}_{j=1}^{\infty} \) is a sequence in \( E \) satisfying that \( \exists p_0 \in cs(E) \) such that for all \( p \in cs(E) \) there is \( C_p > 0 \) with:

(i) \( \sum_{j=1}^{\infty} |x'_j(x)|p(x_j - y_j) \leq p_0(x)C_p \) for each \( x \in E \) and

(ii) \( C_{p_0} \) can be chosen strictly smaller than 1,
**Theorem**

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(ii) \( C_{p_0} \) can be chosen strictly smaller than 1,

then, there exists \( \{y'_j\} \) a sequence in \( E' \) such that \( \{y'_j\}, \{y_j\} \) is an atomic decomposition for \( E \).
Theorem

Let \( \{x'_j\}, \{x_j\} \) be an atomic decomposition of a complete lcs \( E \). Then, if \( \{y_j\}_{j=1}^\infty \) is a sequence in \( E \) satisfying that \( \exists p_0 \in cs(E) \) such that for all \( p \in cs(E) \) there is \( C_p > 0 \) with:

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then, there exists \( \{y'_j\}_{j=1}^\infty \) a sequence in \( E' \) such that \( \{y'_j\}, \{y_j\} \) is an atomic decomposition for \( E \).

This should be compared with a result of Díaz.
By previous result, observe that, if \( x'_1(x_1) \neq 1 \) the map \( x \to \sum_{j=2}^{\infty} x'_j(x) x_j \) is invertible as 1 is not an eigenvalue of the rank one operator \( x \to x'_1(x) x_1 \).
By previous result, observe that, if \( x'_1(x_1) \neq 1 \) the map \( x \to \sum_{j=2}^{\infty} x'_j(x)x_j \) is invertible as 1 is not an eigenvalue of the rank one operator \( x \to x'_1(x)x_1 \).

Hence there exists \((y'_j)_j \subset E'\) such that \(((y'_j)_j, (x_{j+1})_j)\) is an atomic decomposition.
Example

By previous result, observe that, if $x'_1(x_1) \neq 1$ the map $x \to \sum_{j=2}^{\infty} x'_j(x)x_j$ is invertible as 1 is not an eigenvalue of the rank one operator $x \to x'_1(x)x_1$.

Hence there exists $(y'_j)_j \subset E'$ such that $((y'_j)_j, (x_{j+1})_j)$ is an atomic decomposition.

In that case, we can remove an element and still obtain atomic decompositions.
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2 Atomic Decompositions

3 Duality

4 Unconditional atomic decompositions

5 Example
Given an atomic decomposition \((\{x'_j\}, \{x_j\})\) of \(E\) it is rather natural to ask whether \((\{x_j\}, \{x'_j\})\) is an atomic decomposition of \(E'\).
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**Lemma**

If \((\{x'_j\}, \{x_j\})\) is an atomic decomposition of \(E\), then \((\{x_j\}, \{x'_j\})\) is an atomic decomposition of \((E', \sigma(E', E))\).
Given an atomic decomposition \( (\{x'_j\}, \{x_j\}) \) of \( E \) it is rather natural to ask whether \( (\{x_j\}, \{x'_j\}) \) is an atomic decomposition of \( E' \).

**Lemma**

If \( (\{x'_j\}, \{x_j\}) \) is an atomic decomposition of \( E \), then \( (\{x_j\}, \{x'_j\}) \) is an atomic decomposition of \( (E', \sigma (E', E)) \).

The question we are going to face is under which conditions \( (\{x_j\}, \{x'_j\}) \) is an atomic decomposition of \( (E', \beta (E', E)) \).
We define a linear operator $T_n : E \rightarrow E$ as
$$T_n (x) := \sum_{j=n+1}^{\infty} x'_j (x) x_j.$$ 

**Definition**

An atomic decomposition $(\{x'_j\}, \{x_j\})$ is *shrinking* if for all $x' \in E'$,
$$\lim_{n \rightarrow \infty} x' \circ T_n = 0$$
uniformly on the bounded subsets of $E$. 
Shrinking Atomic Decompositions

We define a linear operator \( T_n : E \rightarrow E \) as \( T_n(x) := \sum_{j=n+1}^{\infty} x' \cdot (x) \cdot x_j \).

**Definition**

An atomic decomposition \((\{x'_j\}, \{x_j\})\) is shrinking if for all \( x' \in E' \), \( \lim_{n \rightarrow \infty} x' \circ T_n = 0 \) uniformly on the bounded subsets of \( E \).

**Theorem**

The following are equivalent:

1. \((\{x'_j\}, \{x_j\})\) is a shrinking atomic decomposition of \( E \).

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We define a linear operator $T_n : E \to E$ as $T_n(x) := \sum_{j=n+1}^{\infty} x'_j(x)x_j$.

**Definition**

An atomic decomposition $(\{x'_j\}, \{x_j\})$ is *shrinking* if for all $x' \in E'$, $\lim_{n \to \infty} x' \circ T_n = 0$ uniformly on the bounded subsets of $E$.

**Theorem**

The following are equivalent:

1. $(\{x'_j\}, \{x_j\})$ is a shrinking atomic decomposition of $E$.
2. $(\{x_j\}, \{x'_j\})$ is an atomic decomposition for $E'_\beta$. 
We define a linear operator $T_n : E \rightarrow E$ as $T_n (x) := \sum_{j=n+1}^{\infty} x'_j (x) x_j$.

**Definition**

An atomic decomposition $(\{x'_j\}, \{x_j\})$ is *shrinking* if for all $x' \in E'$, $\lim_{n \rightarrow \infty} x' \circ T_n = 0$ uniformly on the bounded subsets of $E$.

**Theorem**

The following are equivalent:

1. $(\{x'_j\}, \{x_j\})$ is a shrinking atomic decomposition of $E$.
2. $(\{x_j\}, \{x'_j\})$ is an atomic decomposition for $E'_\beta$.
3. For all $x' \in E'$, $\sum_{j=1}^{\infty} x' (x_j) x'_j$ is convergent in $E'_\beta$. 
Definition

An atomic decomposition \((\{x'_j\}, \{x_j\})\) is \textit{boundedly complete} if for all \(x'' \in E''\), the series \(\sum_{j=1}^{\infty} x'_j (x'') x_j\) converges in \(E\).
**Definition**

An atomic decomposition \((\{x'_j\}, \{x_j\})\) is *boundedly complete* if for all \(x'' \in E''_\beta\), the series \(\sum_{j=1}^{\infty} x'_j (x'') x_j\) converges in \(E\).

**Proposition**

1. Let \(\{e_j\}_{j=1}^{\infty}\) be a Schauder basis of \(E\); \((\{e'_j\}, \{e_j\})\) is a boundedly complete atomic decomposition if and only if \(\{e_j\}_{j=1}^{\infty}\) is a boundedly complete Schauder basis (i.e. for every \(\{\alpha_j\}_{j=1}^{\infty} \subset K\) such that \((\sum_{j=1}^{k} \alpha_j e_j)_{k}\) is bounded, then \(\sum_{j=1}^{\infty} \alpha_j e_j\) is convergent).
**Definition**

An atomic decomposition \((\{x'_j\}, \{x_j\})\) is *boundedly complete* if for all \(x'' \in \mathbb{E}''\), the series \(\sum_{j=1}^{\infty} x'_j (x'') x_j\) converges in \(\mathbb{E}\).

**Proposition**

1. Let \(\{e_j\}_{j=1}^{\infty}\) be a Schauder basis of \(\mathbb{E}\); \((\{e'_j\}, \{e_j\})\) is a boundedly complete atomic decomposition if and only if \(\{e_j\}_{j=1}^{\infty}\) is a boundedly complete Schauder basis (i.e. for every \(\{\alpha_j\}_{j=1}^{\infty} \subset \mathbb{K}\) such that \((\sum_{j=1}^{k} \alpha_j e_j)_k\) is bounded, then \(\sum_{j=1}^{\infty} \alpha_j e_j\) is convergent).

2. If \((\{x'_j\}, \{x_j\})\) is a boundedly complete atomic decomposition for \(\mathbb{E}\) with \(\mathbb{E}''_{\beta}\) barrelled, then \(\mathbb{E}\) is complemented in its bidual \(\mathbb{E}''_{\beta}\).
**Definition**

An atomic decomposition \((\{x_j'\}, \{x_j\})\) is *boundedly complete* if for all \(x'' \in E''_\beta\), the series \(\sum_{j=1}^{\infty} x_j' (x'')(x_j)\) converges in \(E\).

**Proposition 1**

Let \(\{e_j\}_{j=1}^{\infty}\) be a Schauder basis of \(E\); \((\{e_j'\}, \{e_j\})\) is a boundedly complete atomic decomposition if and only if \(\{e_j\}_{j=1}^{\infty}\) is a boundedly complete Schauder basis (i.e. for every \(\{\alpha_j\}_{j=1}^{\infty} \subset K\) such that \((\sum_{j=1}^{k} \alpha_j e_j)\) is bounded, then \(\sum_{j=1}^{\infty} \alpha_j e_j\) is convergent).

**Proposition 2**

If \((\{x_j'\}, \{x_j\})\) is a boundedly complete atomic decomposition for \(E\) with \(E''_\beta\) barrelled, then \(E\) is complemented in its bidual \(E''_\beta\).

**Proposition**

If \((\{x_j'\}, \{x_j\})\) is shrinking and boundedly complete atomic decomposition for \(E\), then \(E\) is reflexive.
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Definition

An atomic decomposition \((\{x'_j\}, \{x_j\})\) for a lcs \(E\) is said to be *unconditional* if for every \(x \in E\) we have \(x = \sum_{j=1}^{\infty} x'_j(x) x_j\) with unconditional convergence.
Definition

An atomic decomposition \((\{x'_j\}, \{x_j\})\) for a lcs \(E\) is said to be \textit{unconditional} if for every \(x \in E\) we have \(x = \sum_{j=1}^{\infty} x'_j(x) x_j\) with unconditional convergence.

\textbf{McArthur, Retherford, 1969}

If a series \(\sum_{j=1}^{\infty} x_j\) converges unconditionally, then, for every bounded sequence of scalars \(\{a_j\}\), the series \(\sum_{j=1}^{\infty} a_jx_j\) converges and the operator

\[
\ell_\infty \longrightarrow E \\
\{a_j\} \longrightarrow \sum_{j=1}^{\infty} a_jx_j;
\]

is a continuous linear operator.
Theorem

The following are equivalent:

1. \( E \) admits an unconditional atomic decomposition.
Theorem

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1. $E$ admits an unconditional atomic decomposition.

2. $E$ is isomorphic to a complemented subspace of a complete sequence space with the canonical unit vectors as unconditional Schauder basis.
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The following are equivalent:

1. $E$ admits an unconditional atomic decomposition.
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3. $E$ is isomorphic to a complemented subspace of a complete sequence space with unconditional Schauder basis.
James’ type theorems

**Theorem**

Let $E$ be a Fréchet space which admits an unconditional atomic decomposition $(\{x'_j\}, \{x_j\})$. Then, $(\{x'_j\}, \{x_j\})$ is shrinking if and only if $E$ does not contain a copy of $\ell_1$. 
Theorem

Let $E$ be a Fréchet space which admits an unconditional atomic decomposition $(\{x'_j\}, \{x_j\})$. Then, $(\{x'_j\}, \{x_j\})$ is shrinking if and only if $E$ does not contain a copy of $\ell_1$.

- If $(\{x'_j\}, \{x_j\})$ is shrinking, $E'_\beta$ is separable.
Theorem

Let $E$ be a Fréchet space which admits an unconditional atomic decomposition $(\{x'_j\}, \{x_j\})$. Then, $(\{x'_j\}, \{x_j\})$ is shrinking if and only if $E$ does not contain a copy of $\ell_1$.

- If $(\{x'_j\}, \{x_j\})$ is shrinking, $E'_\beta$ is separable. Therefore $E$ contains no subspace isomorphic to $\ell_1$. 

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Theorem

Let $E$ be a Fréchet space which admits an unconditional atomic decomposition $(\{x'_j\}, \{x_j\})$. Then, $(\{x'_j\}, \{x_j\})$ is shrinking if and only if $E$ does not contain a copy of $\ell_1$.

- If $(\{x'_j\}, \{x_j\})$ is shrinking, $E'_\beta$ is separable. Therefore $E$ contains no subspace isomorphic to $\ell_1$.
- We know that $(\{x_j\}, \{x'_j\})$ is an atomic decomposition of $(E', \sigma(E', E))$ and we prove that it is unconditional.
Theorem

Let $E$ be a Fréchet space which admits an unconditional atomic decomposition ($\{x'_j\}, \{x_j\}$). Then, ($\{x'_j\}, \{x_j\}$) is shrinking if and only if $E$ does not contain a copy of $\ell_1$.

- If ($\{x'_j\}, \{x_j\}$) is shrinking, $E'_\beta$ is separable. Therefore $E$ contains no subspace isomorphic to $\ell_1$.
- We know that ($\{x_j\}, \{x'_j\}$) is an atomic decomposition of $(E', \sigma(E', E))$ and we prove that it is unconditional. By Orlicz-Pettis’ Theorem it is $\mu(E', E)$-unconditionally convergent to $x'$. 
Let $E$ be a Fréchet space which admits an unconditional atomic decomposition ($\{x'_j\}, \{x_j\}$). Then, ($\{x'_j\}, \{x_j\}$) is shrinking if and only if $E$ does not contain a copy of $\ell_1$.

- If ($\{x'_j\}, \{x_j\}$) is shrinking, $E'_\beta$ is separable. Therefore $E$ contains no subspace isomorphic to $\ell_1$.
- We know that ($\{x_j\}, \{x'_j\}$) is an atomic decomposition of $(E', \sigma(E', E))$ and we prove that it is unconditional. By Orlicz-Pettis’ Theorem it is $\mu(E', E)$-unconditionally convergent to $x'$. By a result of Bonet and Lindström in 1993, if $E$ does not contain a copy of $\ell_1$ then every $\mu(E', E)$-null sequence in $E'$ is strongly convergent to zero.
Theorem

Let $E$ be a Fréchet space which admits an unconditional atomic decomposition $(\{x'_j\}, \{x_j\})$. Then, $(\{x'_j\}, \{x_j\})$ is shrinking if and only if $E$ does not contain a copy of $\ell_1$.

- If $(\{x'_j\}, \{x_j\})$ is shrinking, $E'_\beta$ is separable. Therefore $E$ contains no subspace isomorphic to $\ell_1$.
- We know that $(\{x_j\}, \{x'_j\})$ is an atomic decomposition of $(E', \sigma(E', E))$ and we prove that it is unconditional. By Orlicz-Pettis’ Theorem it is $\mu(E', E)$-unconditionally convergent to $x'$. By a result of Bonet and Lindström in 1993, if $E$ does not contain a copy of $\ell_1$ then every $\mu(E', E)$-null sequence in $E'$ is strongly convergent to zero.

Theorem

This result can be extended to boundedly retractive $(LF)$-spaces.
Theorem

If \( E \) admits an unconditional atomic decomposition \((\{x'_j\}, \{x_j\})\), then, \((\{x'_j\}, \{x_j\})\) is boundedly complete if and only if \( E \) does not contain a copy of \( c_0 \).
Theorem

If $E$ admits an unconditional atomic decomposition $(\{x'_j\}, \{x_j\})$, then, $(\{x'_j\}, \{x_j\})$ is boundedly complete if and only if $E$ does not contain a copy of $c_0$.

- If we suppose that $E$ contains a copy of $c_0$, there exists a projection $P$ such that $P(E) \cong c_0$. 
James’ type theorems

**Theorem**

If $E$ admits an unconditional atomic decomposition $(\{x'_j\}, \{x_j\})$, then, $(\{x'_j\}, \{x_j\})$ is boundedly complete if and only if $E$ does not contain a copy of $c_0$.

- If we suppose that $E$ contains a copy of $c_0$, there exists a projection $P$ such that $P(E) \simeq c_0$. Then, $c_0$ is complemented in its bidual $l_\infty$, a contradiction.
If $E$ admits an unconditional atomic decomposition $\left(\{x'_j\}, \{x_j\}\right)$, then $\left(\{x'_j\}, \{x_j\}\right)$ is boundedly complete if and only if $E$ does not contain a copy of $c_0$. 

- If we suppose that $E$ contains a copy of $c_0$, there exists a projection $P$ such that $P(E) \simeq c_0$. Then, $c_0$ is complemented in its bidual $l_\infty$, a contradiction.

- Suppose that $(\{x'_j\}, \{x_j\})$ is not boundedly complete,
Theorem

If $E$ admits an unconditional atomic decomposition ($(x'_j, x_j)$), then, $(x'_j, x_j)$ is boundedly complete if and only if $E$ does not contain a copy of $c_0$.

- If we suppose that $E$ contains a copy of $c_0$, there exists a projection $P$ such that $P(E) \simeq c_0$. Then, $c_0$ is complemented in its bidual $l_\infty$, a contradiction.
- Suppose that $(x'_j, x_j)$ is not boundedly complete, then there exists $U$, a 0-neighborhood, and a sequence $(y_j)$ such that $p_U(y_j) \geq 1$ and $(y_j)_j$ converges to 0 in the topology $\sigma(E, E')$, a contradiction.
Outline

1 Introduction

2 Atomic Decompositions

3 Duality

4 Unconditional atomic decompositions

5 Example
Let $K$ be a compact set in $\mathbb{R}^p$, $p \geq 1$, with $\overset{\circ}{K} \neq \emptyset$ such that $K = \overline{\overset{\circ}{K}}$. 

An atomic decomposition on the space $C^\infty(K)$

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Let $C^\infty(K) := \{ f \in C^\infty(\overset{\circ}{K}) : f$ and all its partial derivatives admit continuous extension to $K \}$. 

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The Fréchet space topology in $C^\infty (K)$ is defined by the seminorms:

$$q_n (f) := \sup \left\{ \left| f^{(\alpha)} (x) \right| : x \in K, |\alpha| \leq n \right\}, n \in \mathbb{N}_0.$$
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**Remark**

No system of exponentials can be a basis in $C^\infty ([0,1])$. 


An atomic decomposition on the space $C^\infty(K)$

**Theorem**

Let us assume that there exists a continuous linear extension operator $T : C^\infty(K) \to C^\infty(\mathbb{R}^p)$. Then there are sequences $(\lambda^j) \subset \mathbb{R}^p$ and $(u_j) \in C^\infty(K)'$ such that $(\{u_j\}, \{e^{2\pi i x \cdot \lambda^j}\})$ is an atomic decomposition for $C^\infty(K)$. 

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The atomic decomposition of $C^\infty(K)$ is shrinking and boundedly complete since $C^\infty(K)$ is a Montel space.
An atomic decomposition on the space $C^\infty (K)$

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Thanks for your attention