Unconditional atomic decompositions in Fréchet spaces

XI Encuentro de Análisis Funcional Alicante - Murcia - València

Alicante

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Atomic Decompositions have been investigated by **Casazza, Gröchenig, Carando, Lassalle, Schmidberg, Korobeĭnik, Taskinen and others**.

Outline

Definition

Let E be a Hausdorff locally convex space.

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Let $\{x_j\}_{j=1}^{\infty} \subset E$ and let $\{x_j'\}_{j=1}^{\infty} \subset E'$, we say that $(\{x_j'\}, \{x_j\})$ is an *atomic decomposition of E* if

$$x = \sum_{j=1}^{\infty} x_j'(x)\, x_j, \qquad \text{for all } x \in E,$$

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the series converging in E.

We denote by ω the space $\mathbb{K}^{\mathbb{N}}$ endowed by the product topology.
A sequence space is a lcs Λ such that $\mathbb{K}^{(\mathbb{N})} \subset \Lambda \subset \omega$, this last inclusion being continuous.

Examples

Example (Leont'ev, 1970's)

For every convex bounded set, $\Omega \subset \mathbb{C}$, there exists a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{C}$ such that, for every $f \in \mathcal{H}(\Omega)$,

$$f(z) = \sum_{j=1}^{\infty} c_j e^{x_j z}$$

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Korobeĭnik, Y. F. and Melikhov, S. N. proved that, if the boundary of Ω is C^2, there exist $\{c_j\}_{j=1}^{\infty}$ depending continuously of f (i.e. $c_j := u_j(f)$ where u_j is a linear and continuous operator). Therefore, we obtain an atomic decomposition.

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Let E be a lcs with a Schauder basis $\{e_j\}_{j=1}^{\infty} \subset E$ and denote by $\{e_j'\}_{j=1}^{\infty} \subset E'$ the functional coefficients. Then $(\{e_j'\}, \{e_j\})$ is an atomic decomposition for E such that $e_j'(e_i) = \delta_{j,i}$ for all $i, j \in \mathbb{N}$.

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Example

Let E be a lcs and let $P : E \to E$ be a continuous linear projection. If $(\{x_j'\}, \{x_j\})$ is an atomic decomposition for E, then $(\{P'(x_j')\}, \{P(x_j)\})$ is an atomic decomposition for $P(E)$.

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From now on E always be a barrelled and complete Hausdorff locally convex space.

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Theorem

A Fréchet space E admits an atomic decomposition if, and only if, E has the bounded approximation property.

Sketch of the Proof

1) $\Rightarrow$ 2)

- We define a injective and continuous linear linear map

$$\begin{array}{rcl} U : E & \longrightarrow & \Lambda \\ x & \longrightarrow & U(x) := \left(x_j'(x)\right)_j . \end{array}$$

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Where $\bigwedge := \{\alpha = (\alpha_j)_j \in \omega : \sum_{j=1}^{\infty} \alpha_j x_j$ is convergent in $E\}$ is a sequence space endowed with the system of seminorms

$$\mathcal{Q} := \left\{ q_p((\alpha_j)_j) := \sup_n p(\sum_{j=1}^{n} \alpha_j x_j), \text{ for all } p \in cs(E) \right\},$$

such that $(\bigwedge, \mathcal{Q})$ is complete.

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- And $S : \bigwedge \longrightarrow E,\ S((\alpha_j)_j) := \sum_{j=1}^{\infty} \alpha_j x_j$, is a continuous linear map.

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- And $S : \bigwedge \longrightarrow E,\ S((\alpha_j)_j) := \sum_{j=1}^{\infty} \alpha_j x_j$, is a continuous linear map.
- From $S \circ U = I_E$ we conclude that U is an isomorphism into its range $U(E)$ and $U \circ S$ is a projection of $\bigwedge$ onto $U(E)$.

Perturbation Result

Theorem

Let $(\{x_j'\}, \{x_j\})$ be an atomic decomposition of a complete lcs E. Then, if $\{y_j\}_{j=1}^{\infty}$ is a sequence in E satisfying that $\exists p_0 \in cs(E)$ such that for all $p \in cs(E)$ there is $C_p > 0$ with:
(i) $\sum_{j=1}^{\infty} |x_j'(x)| p(x_j - y_j) \leq p_0(x) C_p$ for each $x \in E$ and
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This should be compared with a result of *Díaz*.

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By previous result, observe that, if $x_1'(x_1) \neq 1$ the map $x \to \sum_{j=2}^{\infty} x_j'(x)x_j$ is invertible as 1 is not an eigenvalue of the rank one operator $x \to x_1'(x)x_1$.

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By previous result, observe that, if $x'_1(x_1) \neq 1$ the map $x \to \sum_{j=2}^{\infty} x'_j(x)x_j$ is invertible as 1 is not an eigenvalue of the rank one operator $x \to x'_1(x)x_1$.

Hence there exists $(y'_j)_j \subset E'$ such that $((y'_j)_j, (x_{j+1})_j)$ is an atomic decomposition.

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Hence there exists $(y_j')_j \subset E'$ such that $((y_j')_j, (x_{j+1})_j)$ is an atomic decomposition.

In that case, we can remove an element and still obtain atomic decompositions.

Outline

Introduction

Given an atomic decomposition $(\{x_j'\}, \{x_j\})$ of E it is rather natural to ask whether $(\{x_j\}, \{x_j'\})$ is an atomic decomposition of E'.

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Lemma

If $(\{x_j'\},\{x_j\})$ is an atomic decomposition of E, then $(\{x_j\},\{x_j'\})$ is an atomic decomposition of $(E',\sigma(E',E))$.

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Lemma

If $(\{x_j'\}, \{x_j\})$ is an atomic decomposition of E, then $(\{x_j\}, \{x_j'\})$ is an atomic decomposition of $(E', \sigma(E', E))$.

The question we are going to face is under which conditions $(\{x_j\}, \{x_j'\})$ is an atomic decomposition of $(E', \beta(E', E))$.

Shrinking Atomic Decompositions

We define a linear operator $T_n : E \to E$ as $T_n(x) := \sum_{j=n+1}^{\infty} x_j'(x)\, x_j$.

Definition

An atomic decomposition $(\{x_j'\}, \{x_j\})$ is *shrinking* if for all $x' \in E'$, $\lim_{n\to\infty} x' \circ T_n = 0$ uniformly on the bounded subsets of E.

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Theorem

The following are equivalent:

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The following are equivalent:

1. $(\{x_j'\}, \{x_j\})$ is a shrinking atomic decomposition of E.
2. $(\{x_j\}, \{x_j'\})$ is an atomic decomposition for E'_β.
3. For all $x' \in E'$, $\sum_{j=1}^{\infty} x'(x_j)\, x_j'$ is convergent in E'_β.

Boundedly Complete Atomic Decompositions

Definition

An atomic decomposition $(\{x_j'\}, \{x_j\})$ is *boundedly complete* if for all $x'' \in E_\beta''$, the series $\sum_{j=1}^{\infty} x_j'(x'')\, x_j$ converges in E.

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Proposition

1. Let $\{e_j\}_{j=1}^{\infty}$ be a Schauder basis of E; $(\{e_j'\}, \{e_j\})$ is a boundedly complete atomic decomposition if and only if $\{e_j\}_{j=1}^{\infty}$ is a boundedly complete Schauder basis (i.e. for every $\{\alpha_j\}_{j=1}^{\infty} \subset \mathbb{K}$ such that $(\sum_{j=1}^{k} \alpha_j e_j)_k$ is bounded, then $\sum_{j=1}^{\infty} \alpha_j e_j$ is convergent).

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2. If $(\{x_j'\}, \{x_j\})$ is a boundedly complete atomic decomposition for E with E_β'' barrelled, then E is complemented in its bidual E_β''.

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Proposition

If $(\{x_j'\}, \{x_j\})$ is shrinking and boundedly complete atomic decomposition for E, then E is reflexive.

Outline

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An atomic decomposition $(\{x_j'\}, \{x_j\})$ for a lcs E is said to be *unconditional* if for every $x \in E$ we have $x = \sum_{j=1}^{\infty} x_j'(x)\, x_j$ with unconditional convergence.

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McArthur, Retherford, 1969

If a series $\sum_{j=1}^{\infty} x_j$ converges unconditionally, then, for every bounded sequence of scalars $\{a_j\}$, the series $\sum_{j=1}^{\infty} a_j x_j$ converges and the operator

$$\begin{array}{ccc} \ell_\infty & \longrightarrow & E \\ \{a_j\} & \longrightarrow & \sum_{j=1}^{\infty} a_j x_j; \end{array}$$

is a continuous linear operator.

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James' type theorems

Theorem

Let E be a Fréchet space which admits an unconditional atomic decomposition $(\{x_j'\}, \{x_j\})$. Then, $(\{x_j'\}, \{x_j\})$ is shrinking if and only if E does not contain a copy of ℓ_1.

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- If $(\{x_j'\}, \{x_j\})$ is shrinking, E'_β is separable. Therefore E contains no subspace isomorphic to ℓ_1.
- We know that $(\{x_j\}, \{x_j'\})$ is an atomic decomposition of $(E', \sigma(E', E))$ and we prove that it is unconditional.

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- If $(\{x_j'\},\{x_j\})$ is shrinking, E'_β is separable. Therefore E contains no subspace isomorphic to ℓ_1.
- We know that $(\{x_j\},\{x_j'\})$ is an atomic decomposition of $(E',\sigma(E',E))$ and we prove that it is unconditional. By Orlicz-Pettis' Theorem it is $\mu(E',E)$-unconditionally convergent to x'. By a result of Bonet and Lindström in 1993, if E does not contain a copy of ℓ_1 then every $\mu(E',E)$-null sequence in E' is strongly convergent to zero.

Theorem

This result can be extended to boundedly retractive (LF)-spaces.

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Theorem

If E admits an unconditional atomic decomposition $(\{x_j'\}, \{x_j\})$, then, $(\{x_j'\}, \{x_j\})$ is boundedly complete if and only if E does not contain a copy of c_0.

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Theorem

If E admits an unconditional atomic decomposition $(\{x_j'\}, \{x_j\})$, then, $(\{x_j'\}, \{x_j\})$ is boundedly complete if and only if E does not contain a copy of c_0.

- If we suppose that E contains a copy of c_0, there exists a projection P such that $P(E) \simeq c_0$.

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- If we suppose that E contains a copy of c_0, there exists a projection P such that $P(E) \simeq c_0$. Then, c_0 is complemented in its bidual l_∞, a contradiction.

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- Suppose that $(\{x_j'\},\{x_j\})$ is not boundedly complete,

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- If we suppose that E contains a copy of c_0, there exists a projection P such that $P(E) \simeq c_0$. Then, c_0 is complemented in its bidual l_∞, a contradiction.
- Suppose that $(\{x_j'\},\{x_j\})$ is not boundedly complete, then there exists U, a 0-neighborhood, and a sequence $\{y_j\}$ such that $p_U(y_j) \geq 1$ and $(y_j)_j$ converges to 0 in the topology $\sigma(E, E')$, a contradiction.

Outline

An atomic decomposition on the space $C^\infty(K)$

Let K be a compact set in $\mathbb{R}^p$, $p \geq 1$, with $\overset{\circ}{K} \neq \emptyset$ such that $K = \overline{\overset{\circ}{K}}$.

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Let $C^\infty(K) := \{f \in C^\infty(\overset{\circ}{K}) : f$ and all its partial derivatives admit continuous extension to $K\}$.

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The Fréchet space topology in $C^\infty(K)$ is defined by the seminorms:

$$q_n(f) := \sup\left\{\left|f^{(\alpha)}(x)\right| : x \in K,\ |\alpha| \leq n\right\}, n \in \mathbb{N}_0.$$

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Remark

No system of exponentials can be a basis in $C^\infty([0,1])$.

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Theorem

Let us assume that there exists a continuous linear extension operator $T : C^{\infty}(K) \to C^{\infty}(\mathbb{R}^p)$. Then there are sequences $\left(\lambda^j\right) \subset \mathbb{R}^p$ and $(u_j) \in C^{\infty}(K)'$ such that $(\{u_j\}, \{e^{2\pi i x \cdot \lambda^j}\})$ is an atomic decomposition for $C^{\infty}(K)$.

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- Let $M > 0$ such that $K \subset [-M, M]^p$
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The atomic decomposition of $C^{\infty}(K)$ is shrinking and boundedly complete since $C^{\infty}(K)$ is a Montel space.

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Thanks for your attention