Some fixed point theorems in fuzzy metric spaces from Banach's principle

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Here we present the concept of p-metric that use to obtain some well-known fixed point theorems in fuzzy metric spaces from the classical Banach's principle

$$a \oplus_1 b = \min\{1, a + b\}$$

$$\oplus_p : [0,1] \times [0,1] :\to [0,1]$$

$$\oplus_p(a,b) = \min\{1, (a^p + b^p)^{1/p}\},\ p > 0.$$

p-sums (Yager t-conorms)

$$(X, d) \to (X, d_1), \text{ where } d_1 = \min\{1, d\}$$

$$\tau_d \equiv \tau_{d_1}$$

Definition of p-metric space

A p-metric space is a triple $(X, D, \oplus_p)$ such that X is a nonempty set, $\oplus_p$ is a p-sum and D is a fuzzy set in $X \times X$ such that for all $x, y, z \in X$:
(i) $D(x, y) = 0$ if and only if $x = y$
(ii) $D(x, y) = D(y, x)$
(iii) $D(x, z) \leq D(x, y) \oplus_p D(y, z)$

We will say that $(D, \oplus_p)$ is a p-metric on X.

For each $x \in X$ and $r > 0$ we can define the open ball $B_D(x, r) = \{y \in X : D(x, y) < r\}$ and it is obvious that $B_D(x, r_1) \subseteq B_D(x, r_2)$ provided that $r_1 \leq r_2$. Consequently, we may define a topology τ_D on X as $\tau_D = \{A \subseteq X : \text{for each } x \in A \text{ there exists } r > 0 \text{ such that } B_D(x, r) \subseteq A\}$.

If $(D, \oplus_p)$ is a p-metric on X then $d(x,y) = D^p(x,y)$ for all $x, y \in X$ is a metric on X and that $\tau_d = \tau_D$. Reciprocally, if d is a $1-$bounded metric on X then $D(x,y) = d^{1/p}(x,y)$ for all $x, y \in X$ is a p-metric on X for $\oplus_p$, $p > 0$.

t-norms and t-conorms

A t-norm is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:(i) $*$ is associative and commutative; (ii) $a * 1 = a$ for every $a \in [0,1]$, (iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0,1]$.

A t-conorm is a binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:(i) $\diamond$ is associative and commutative; (ii) $a \diamond 0 = a$ for every $a \in [0,1]$, (iii) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0,1]$.

If $*$ is a (continuous) t-norm we can define a (continuous) t-conom $\diamond_*$as follows: $a \diamond_* b = 1 - [(1-a) * (1-b)]$ for all $a, b \in [0,1]$

p-sums are continuous t-conorms (Yager continuous t-conorms)

Zadeh. L, Similarity relations and fuzzy orderings. Information Sciences, 3, 159-176, 1971.

Definition of similarity relation

A similarity relation on a set X is a pair $(E, *)$ such that $*$ is a t-norm and E is a fuzzy set in $X \times X$ such that for all $x, y, z \in X$:

(E1) $E(x, y) = 1$ if and only if $x = y$

(E2) $E(x, y) = E(y, x)$

(E3) $E(x, z) \geq E(x, y) * E(y, z)$.

If we define $D(x, y) = 1 - E(x, y)$ for all $x, y \in X$, then $D(x, z) \leq D(x, y) \diamond_* D(y, z)$. So if $\diamond_* \leq \oplus_p$ for some $p > 0$ then $(X, D, \oplus_p)$ is a p-metric space.

I. Kramosil and J. Michalek, Fuzzy metrics and Statistical metric spaces, Kibernetica v.2 n 2, 1975

Definition

A fuzzy metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, \infty)$ such that for all $x, y, z \in X$:

(FM1) $M(x, y, 0) = 0$;

(FM2) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;

(FM3) $M(x, y, t) = M(y, x, t)$;

(FM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$;

(FM5) $M(x, y, \_) : \mathbb{R}^+ \to [0, 1]$ is left continuous.

By a fuzzy metric space we mean a triple $(X, M, *)$ such that X is a set and $(M, *)$ is a fuzzy metric on X.

Given a fuzzy metric $(M,*)$ on a set X we can define a open ball for each $x \in X$, $t > 0$ and $\varepsilon \in (0,1)$ as $B_M(x,\varepsilon,t) = \{y \in X : M(x,y,t) > 1-\varepsilon\}$. Consequently, we may define a topology τ_M on X as $\tau_M = \{A \subseteq X : \text{for each } x \in A \text{ there exists } \varepsilon \in (0,1) \text{ and } t > 0 \text{ such that } B_M(x,\varepsilon,t) \subseteq A\}$.

A. George, P. Veeramani, On some results in fuzzy metric spaces. Fuzzy Sets and Systems, 64 (1994), 395-399.

Definition of Cauchy sequence

A Cauchy sequence in a fuzzy metric space $(X, M, *)$ is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that for each $\varepsilon \in (0,1)$ and $t > 0$ there exists an $n_0 \in \mathbb{N}$ satisfying $M(x_n, x_m, t) > 1 - \varepsilon$ whenever $n, m \geq n_0$.

Definition of complete fuzzy metric space

A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ converges with respect to the topology τ_M, i.e, if there exists $y \in X$ such that for each $t > 0$, $\lim_n M(y, x_n, t) = 1$.

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V. Gregori, S. Romaguera, Some properties of fuzzy metric spaces. Fuzzy Sets and Systems, 115 (2000), 485-489.

V. Gregori, S. Romaguera, Characterizing completable fuzzy metric spaces, Fuzzy Sets and Systems, 144 (2004), 411-420.

Definition of stationary fuzzy metric space

A fuzzy metric space $(X, M, *)$ is said to be stationary if M does not depend on t

A similarity relation $(E, *)$ on a set X is a stationary fuzzy metric space $(X, E, *)$ by defining $E(x, y, 0) = 0$ when $*$ is continuous

If $(X, M, *)$ is a stationary fuzzy metric space such that $\diamond_* \leq \oplus_p$ for some $p > 0$ then $(X, D, \oplus_p)$ is a p-metric space where $D(x, y) = 1 - M(x, y)$ for all $x, y \in X$. Reciprocally, if $(X, D, \oplus_p)$ is a p-metric space then $(X, M, *)$ is a stationary fuzzy metric space, where $M(x, y) = 1 - D(x, y)$ for all $x, y \in X$, $M(x, y, 0) = 0$ and $\diamond_* \leq \oplus_p$.

If$(X, M, *)$ is a fuzzy metric space such that $\diamond_* \leq \oplus_p$ for some $p > 0$ then $d(x,y) = D^p(x,y) = (1 - M(x,y))^p$ for all $x, y \in X$ is a metric on X and that $\tau_d = \tau_D = \tau_M$.In particular if $\diamond_* \leq \oplus_1$ then $d(x,y) = 1 - M(x,y)$ is a metric on X.

V. Radu, On the triangle inequality in PM-spaces. STPA, West University of Timisoara 39 (1978)

Theorem 1

Let $(X, M, *)$ be a fuzzy metric space such that $\diamond_* \leq \oplus_1$. For each $x, y \in X$ put

$$d_R(x, y) = \sup\{t \geq 0 : 1 - M(x, y, t) \geq t\}.$$

Then d_R is a metric on X such that

$$d_R(x, y) < \varepsilon \Leftrightarrow M(x, y, \varepsilon) > 1 - \varepsilon,$$

for all $\varepsilon \in (0, 1)$.

Therefore, the topologies induced by $(M, *)$ and d_R coincide on X. In particular, $(X, M, *)$ is complete if and only if (X, d_R) is complete.

Example

Let $(X, M, *)$ be a fuzzy metric space such that $\diamond_* \leq \oplus_p$ for some $p \in (0,1)$. The function $d : X \times X \to \mathbb{R}^+$, defined as

$d(x,y) = \sup\{t \geq 0 : (1 - M(x,y,t))^p \geq t\}$

is a metric on X such that

$d(x,y) < \varepsilon \Leftrightarrow M(x,y,\varepsilon) > 1 - \varepsilon^{1/p}$,

for all $\varepsilon \in (0,1)$.

Therefore, the topologies induced by $(M, *)$ and d coincide on X. In particular, $(X, M, *)$ is complete if and only if (X, d) is complete.

F.Castro-Company, S. Romaguera and P. Tirado, On the construction of metrics from fuzzy metrics and its application to the fixed point theory of multivalued mappings, Fixed Point Theory and Applications (2015) 2015:226.

Theorem 2

Let $(X, M, *)$ be a fuzzy metric space. Suppose that there exists a function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(c1) α is strictly increasing on $[0, 1]$;
(c2) $0 < \alpha(t) \leq t$ for all $t \in (0, 1)$ and $\alpha(t) > 1$ for all $t > 1$;
(c3) $\alpha(t+s) \geq \alpha(t) \diamond_* \alpha(s)$;

Then the function $d_\alpha : X \times X \to \mathbb{R}^+$ defined as

$$d_\alpha(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - \alpha(t)\},$$

is a metric on X such that $d_\alpha(x, y) \leq 1$ for all $x, y \in X$.

If, in addition, the function α is left continuous on $(0,1]$, then

$$d_\alpha(x,y) < \varepsilon \Leftrightarrow M(x,y,\varepsilon) > 1 - \alpha(\varepsilon),$$

for all $\varepsilon \in (0,1)$. Thus the topologies induced by $(M,*)$ and d_α coincide on X. Moreover, $(X,M,*)$ is complete if and only if (X,d_α) is complete.

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F. Castro-Company, P. Tirado, On Yager and Hamacher t-norms and fuzzy metric spaces, International Journal of Intelligent systems, 29 (2014), 1179-1180.

Theorem 3

Let $(X, M, *)$ be a complete fuzzy metric space such that $\diamond_* \leq \oplus_p$ for some $p > 0$. If T is a self-map on X such that there is $k \in (0,1)$ satisfying

$$M(Tx, Ty, t) \geq 1 - k + kM(x, y, t)$$

for all $x, y \in X$ and $t > 0$, then T has a unique fixed point.

$1 - M(Tx, Ty, t) \leq k(1 - M(x, y, t))$
$\Leftrightarrow [1 - M(Tx, Ty, t)]^p \leq [k(1 - M(x, y, t))]^p$
$\Leftrightarrow [1 - M(Tx, Ty, t)]^p \leq k^p[(1 - M(x, y, t))]^p$. So we can write

$\sup\{t \geq 0 : (1 - M(Tx, Ty, t))^p \geq t\} \leq k^p \sup\{t \geq 0 : (1 - M(x, y, t))^p \geq t\}$

i.e, following the notation in Example 1

$d(Tx, Ty) \leq k^p d(x, y)$. Since (X, d) is complete, by the Banach contraction principle T has a unique fixed point.

A. George, P. Veeramani, On some results in fuzzy metric spaces. Fuzzy Sets and Systems, 64 (1994), 395-399

Definition GV-fuzzy metric

A GV-fuzzy metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times (0, \infty)$ such that for all $x, y, z \in X$ and $t, s > 0$:

(GV1) $M(x, y, t) > 0$;

(GV2) $x = y$ if and only if $M(x, y, t) = 1$;

(GV3) $M(x, y, t) = M(y, x, t)$;

(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;

(GV5) $M(x, y, \_) : (0, \infty) \to (0, 1]$ is continuous.

It is interesting to remark the fact that every GV-fuzzy metric space $(X, M, *)$ can be considered as a fuzzy metric space in the sense of Kramosil and Michalek, simply putting $M(x, y, 0) = 0$ for all $x, y \in X$, so the previous results remain valid for GV-fuzzy metric spaces.

V. Gregori, A. Sapena, On fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 125 (2002), 245-253

fuzzy-contractive selp-map

Let $(X, M, *)$ be a GV-fuzzy metric space and $T : X \to X$ a self-map. We will say that T is fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k(\frac{1}{M(x, y, t)} - 1)$$

for all $x, y \in X$ and $t > 0$.

Theorem 4

Let $(X, M, \wedge)$ be a complete GV-fuzzy metric space.Then every fuzzy contractive self-map T on X has a unique fixed point.

Let $\alpha(t) = \frac{t}{t+1}$ for $t \in [0,1]$ and 2 for $t > 1$, then α satisties the conditions of Theorem 2, then the function $d_\alpha : X \times X \to \mathbb{R}^+$ defined as

$d_\alpha(x,y) = \sup\{t \geq 0 : M(x,y,t) \leq 1 - \frac{t}{t+1}\}$, or , equivalently
$d_\alpha(x,y) = \sup\{t \geq 0 : \frac{1}{M(x,y,t)} - 1 \geq t\}$

is a metric on X , thus (X, d_α) is a complete metric space.

Let T a fuzzy contractive self-map, then

$$\frac{1}{M(Tx,Ty,t)} - 1 \leq k(\frac{1}{M(x,y,t)} - 1)$$

so, we have

$$\sup\{t \geq 0 : \frac{1}{M(Tx,Ty,t)} - 1 \geq t\} \leq k \sup\{t \geq 0 : \frac{1}{M(x,y,t)} - 1 \geq t\},$$ i.e

$d_\alpha(Tx, Ty) \leq k d_\alpha(x, y)$. By the Banach contraction principle, T has a unique fixed point.

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