The q-hyperconvex hull of a T_0-quasi-metric space

Hans-Peter A. Künzi
University of Cape Town

July 20, 2014

Table of contents

Metric spaces

Definition

A **metric** *m on a set X is a function* $m : X \times X \to [0, \infty)$ *satisfying the conditions:*
(a) For all $x, y \in X$, $m(x, y) = 0$ *if and only if* $x = y$.
(b) $m(x, y) = m(y, x)$ *whenever* $x, y \in X$.
(c) $m(x, z) \leq m(x, y) + m(y, z)$ *whenever* $x, y, z \in X$.
(Here $[0, \infty)$ *denotes the set of the nonnegative reals.)*

Injective metric spaces

Definition

A map $f:(X,d)\to(Y,e)$ between metric spaces (X,d) and (Y,e) is called **isometric** *provided that $d(x,y)=e(f(x),f(y))$ whenever $x,y\in X$.*

A map $f:(X,d)\to(Y,e)$ between metric spaces (X,d) and (Y,e) is called **nonexpansive** *provided that $e(f(x),f(y))\leq d(x,y)$ whenever $x,y\in X$.*

A metric space (X,m) is said to be **injective** *if it has the following extension property for nonexpansive maps: Whenever Y is a subspace of a metric space Z and $f:Y\to X$ is a nonexpansive map, then f has a nonexpansive extension $\widetilde{f}:Z\to X$.*

Hyperconvexity

Definition

A metric space (X, m) is called **hyperconvex** *if for each $A \subseteq X$ and each family of positive real numbers $(r_x)_{x \in A}$ the conditions $m(x, y) \leq r_x + r_y$ whenever $x, y \in A$ imply that $\emptyset \neq \bigcap_{x \in A} C_m(x, r_x)$.*
Here $C_m(x, r_x)$ denotes the closed ball of radius r_x at $x \in A$.

Proposition

(1956: N. Aronszajn and P. Panitchpakdi) A metric space is hyperconvex if and only if it is injective.

Convexity

Definition

Let (X, m) be a metric space. Then X is **metrically convex** *if for any point $x, y \in X$ and positive numbers r and s such that $m(x, z) \leq r + s$, there exists $y \in X$ such that $m(x, y) \leq r$ and $m(y, z) \leq s$.*

Example

ℓ_∞ is hyperconvex. This is the space whose elements consist of all bounded sequences $(x_n)_{n\in\mathbb{N}}$ of real numbers, with distance $m_\infty((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$.

Hyperconvex hull (1964: Isbell)

Remark

The **metric hyperconvex hull $\mathbf{M_X}$** *of a metric space* (X, m) *consists of all the minimal ample functions* $f : X \to [0, \infty)$ *where we call* f **ample** *if* $m(x, y) \leq f(x) + f(y)$ *whenever* $x, y \in X$ *and* f *is called* **minimal** *among the ample functions on* X *if it is minimal with respect to the pointwise order on these functions.*
Then $\mathbf{E}(f, g) = \sup_{x \in X} |f(x) - g(x)|$ *whenever* $f, g \in M_X$ *defines a metric on* M_X.
Furthermore given $x \in X$, $\mathbf{h}(x) = m(x, y)$ *whenever* $y \in X$ *defines an isometric embedding of* (X, m) *into* (M_X, E).
The closure of $h(X)$ *in* M_X *yields the completion of the metric space* (X, m).

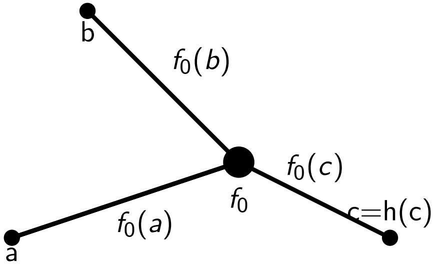


Figure: Consider the metric space (X, m) with 3 points a, b, c. The injective hull of (X, m) is determined by a function f_0 defined as follows: $f_0(a) = \frac{ab+ac-cb}{2}$, $f_0(b) = \frac{bc+ba-ca}{2}$ and $f_0(c) = \frac{ca+cb-ab}{2}$.
Here for instance $ab = m(a, b)$.

Motivation

“Besides, one insists that the distance function be symmetric, that is, $d(x, y) = d(y, x)$. (This unpleasantly limits many applications: the effort of climbing up to the top of a mountain in real life, as well as in mathematics, is not at all the same as descending back to the starting point).“

M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, vol. 152 of Progress in Mathematics, Birkhäuser.

T_0-quasi-metric spaces

Definition

Let X be a set and $d : X \times X \to [0, \infty)$ be a function. Then d is called a **quasi-pseudometric** *on X if*
(a) $d(x, x) = 0$ whenever $x \in X$, and
(b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.
We shall say that (X, d) is a **T_0-quasi-metric space** *provided that d also satisfies the following condition: For each $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$.*

Definition

Given a T_0-quasi-metric space (X, d), the **specialization (partial) order $\leq_d$ of** *d is defined as follows: For each $x, y \in X$, set $x \leq_d y$ if $d(x, y) = 0$.*

Examples of T_0-quasi-metrics

Example

Let $(X, \leq)$ be a partially ordered set. Then the function d on $X \times X$ defined by $d(x, y) = 0$ if $x \leq y$ and $d(x, y) = 1$ otherwise, is called the **natural T_0-quasi-metric** *of the partial order $\leq$ on X.*

Example

Given two real numbers a and b we shall write $\mathbf{a} \dot{-} \mathbf{b}$ *for* $\max\{a - b, 0\}$.
Then $\mathbf{u}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \dot{-} \mathbf{y}$ *with $x, y \in \mathbb{R}$ defines the* **standard** *T_0-quasi-metric on the set $\mathbb{R}$ of the reals.*

The dual and the supremum of a T_0-quasi-metric

Let d be a quasi-pseudometric on a set X. Then $d^{-1}: X \times X \to [0, \infty)$ defined by $\mathbf{d^{-1}}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the **conjugate** or **dual** quasi-pseudometric of d.

If d is a T_0-quasi-metric on X, then $\mathbf{d^s} = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a metric on X.

Given $x \in X$ and a nonnegative real number r we set $\mathbf{C_d(x, r)} = \{y \in X : d(x, y) \leq r\}$.

This set is $\tau(d^{-1})$-closed, where $\tau(\mathbf{d})$ is the topology having the balls $\mathbf{B_d(x}, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ with $x \in X$ and $\epsilon > 0$ as basic (open) sets.

For $(\mathbb{R}, u)$, $x \in \mathbb{R}$ and $\epsilon > 0$ we obtain

$$B_u(x, \epsilon) = (x - \epsilon, \infty),$$

$$C_u(x, \epsilon) = [x - \epsilon, \infty),$$

$$B_{u^{-1}}(x, \epsilon) = (-\infty, x + \epsilon),$$

$$C_{u^{-1}}(x, \epsilon) = (-\infty, x + \epsilon],$$

and

$$B_{u^s}(x, \epsilon) = (x - \epsilon, x + \epsilon),$$

$$C_{u^s}(x, \epsilon) = [x - \epsilon, x + \epsilon].$$

Ample function pairs (Kemajou, Otafudu, etc.)

Let (X, d) be a T_0-quasi-metric space. We shall say that a function pair $f = (f_1, f_2)$ on (X, d) where $f_i : X \to [0, \infty)$ $(i = 1, 2)$ is **ample** provided that $d(x, y) \leq f_2(x) + f_1(y)$ whenever $x, y \in X$. Let $\mathbf{P_X}$ denote the set of all ample function pairs on (X, d). (In such situations we may also write $P_{(X,d)}$ in cases where d is not obvious.) For each $f, g \in P_X$ we set

$$\mathbf{D}(\mathbf{f}, \mathbf{g}) = \sup_{x \in X}(f_1(x) \dot{-} g_1(x)) \vee \sup_{x \in X}(g_2(x) \dot{-} f_2(x)).$$

Then D is an extended quasi-pseudometric on P_X.

We shall call a function pair f **minimal** on (X, d) (among the ample function pairs on (X, d)) if it is ample and whenever g is ample on (X, d) and for each $x \in X$ we have $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$, then $g = f$.

Injective hull (directed span) of a T_0-quasi-metric space

Zorn's Lemma implies that below each ample function pair there is a minimal ample pair (a more constructive method is due to Dress).

By $\mathbf{Q_X}$ we shall denote the set of all minimal ample pairs on (X, d) equipped with the restriction of D to $Q_X \times Q_X$, which we shall also denote by D. Then D is a (real-valued) T_0-quasi-metric on $Q_X \times Q_X$.

For each $x \in X$ we can define the minimal function pair

$$\mathbf{f_x}(y) = (d(x, y), d(y, x))$$

(whenever $y \in X$) on (X, d). The map $\mathbf{e}$ defined by $x \mapsto f_x$ whenever $x \in X$ defines an isometric embedding of (X, d) into (Q_X, D). Then (Q_X, D) is called the q-**hyperconvex hull** of (X, d).

Some important facts

We have $f = (f_1, f_2) \in Q_X$ if and only if the following equations $(*)$ are satisfied:

$$f_1(x) = \sup\{d(y,x) \dot{-} f_2(y) : y \in X\}$$

and

$$f_2(x) = \sup\{d(x,y) \dot{-} f_1(y) : y \in X\}$$

whenever $x \in X$. In particular pairs satisfying these equations are ample on (X, d).

A kind of 'metric' density of $e(X)$ in Q_X :For any $y_1, y_2 \in Q_X$, we have that

$$D(y_1, y_2) =$$

$$\sup\{(D(f_{x_1}, f_{x_2}) - D(f_{x_1}, y_1) - D(y_2, f_{x_2})) \vee 0 : x_1, x_2 \in X\}.$$

Important facts

(1) Interesting case where
$D(f_{x_1}, y_1) + D(y_1, y_2) + D(y_2, f_{x_2}) = D(f_{x_1}, f_{x_2})$ for some $x_1, x_2 \in X$.

(2) $f \in Q_X$ implies that $f_1(x) - f_1(y) \leq d^{-1}(x, y)$ and
$f_2(x) - f_2(y) \leq d(x, y)$ whenever $x, y \in X$.

(3) $\sup_{x\in X}(f_1(x) \dot{-} g_1(x)) = \sup_{x\in X}(g_2(x) \dot{-} f_2(x))$ whenever
$f, g \in Q_X$.

(4) $D(f, f_x) = f_1(x)$ and $D(f_x, f) = f_2(x)$ whenever $x \in X$ and
$f \in Q_X$.

The one-sided approach to the q-hyperconvex hull

The second component f_2 of a minimal ample pair (f_1, f_2) on (X, d) satisfies the following equation $(**)$:

$$f_2(x) = \sup_{y \in X}(d(x,y) \dot{-} \sup_{y' \in X}(d(y',y) \dot{-} f_2(y')))$$

whenever $x \in X$.
Indeed equation $(**)$ characterizes exactly those functions $f : X \to [0, \infty)$ that are second components of minimal ample pairs on (X, d). An analogous result holds for the first components of minimal ample pairs on (X, d).
These facts can be explained by the underlying Isbell conjugation adjunction.

q-hyperconvexity

A T_0-quasi-metric space X is said to be q-**hyperconvex** if $f \in Q_X$ implies that there is an $x \in X$ such that $f = f_x$.

An intrinsic characterization of q-hyperconvexity is the following: A T_0-quasi-metric space (X, d) is q-hyperconvex if and only if, given $A \subseteq X$ and families of nonnegative reals $(r_x)_{x \in A}$ and $(s_x)_{x \in A}$ such that $d(x, y) \leq r_x + s_y$ whenever $x, y \in A$, we have that $\bigcap_{x \in A}(C_d(x, r_x) \cap C_{d^{-1}}(x, s_x)) \neq \emptyset$.

A T_0-quasi-metric space is q-hyperconvex if and only if it is injective in the category of T_0-quasi-metric spaces (and nonexpansive maps).

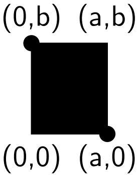
An explicit example

Example

Let $a, b \in [0, \infty)$ *be such that* $a + b \neq 0$ *and let* $Y = [0, a] \times [0, b]$. *Set*

$$D((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = (\alpha_1 \dot{-} \beta_1) \vee (\alpha_2 \dot{-} \beta_2)$$

whenever $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in Y$. *Then* Y *can be identified with the q-hyperconvex hull of the* T_0*-quasi-metric subspace* $X = \{(a, 0), (0, b)\}$ *of* Y.



Example

(The general quasi-metric 'segment' I_{ab}.) Let $X = [0,1]$. Choose $a, b \in [0, \infty)$ such that $a + b \neq 0$. Set $d_{ab}(x,y) = (x-y)a$ if $x > y$ and $d_{ab}(x,y) = (y-x)b$ if $y \geq x$. Then $([0,1], d_{ab})$ is a T_0-quasi-metric space.

Let (X,d) be a T_0-quasi-metric space and $f, g \in Q_X$ with $f \neq g$. Set $a = D(f,g)$ and $b = D(g,f)$. Then there is an isometric embedding $([0,1], d_{ab}) \to (Q_X, D)$ connecting g to f.

If we equip the unit interval $[0,1]$ with the restriction of $\tau(u^s)$ and Q_X with the topology $\tau(D)$, then Q_X is contractible in the classical sense.

Tight extensions (1984: Dress, for the metric case)

Let X be a subspace of a T_0-quasi-metric space (Y, d). Then Y is called a **tight extension** of X if for any quasi-pseudometric e on Y that satisfies $e \leq d$ and agrees with d on $X \times X$ we have $e = d$.

Proposition

For any T_0-quasi-metric space (X, d) the q-hyperconvex hull Q_X is a (maximal) tight extension of $e(X)$.

Q_X for metric space X

A nonempty partially ordered set X is called a **complete lattice** if $\bigvee S$ and $\bigwedge S$ exist for any subset $S \subseteq X$.

Example

The T_0-quasi-metric space $(\mathbb{R}, u)$ is q-hyperconvex. The specialization order $\leq$ of that space is the standard order on $\mathbb{R}$; hence $(\mathbb{R}, \leq)$ is not a complete lattice.

$(\mathbb{R}, u^s)$ is not q-hyperconvex. ($(\mathbb{R}^2, u \times u^{-1})$ is the q-hyperconvex hull of its diagonal.)
(Willerton) The hyperconvex hull of a metric space X is isometric to the largest metric subspace containing $e(X)$ in the q-hyperconvex hull of X.

Endpoints in a T_0-quasi-metric space (Isbell; Haihambo, Agyingi, etc.)

Definition

Let (X, d) be a quasi-pseudometric space.
(a) A finite sequence $(x_1, x_2, \ldots, x_n)$ in X is called **collinear** *in (X, d) provided that $i < j < k \leq n$ implies that $d(x_i, x_k) = d(x_i, x_j) + d(x_j, x_k)$.*
(b) An element $x \in X$ is called an **endpoint** *of (X, d) provided that there exists an element y in (X, d) such that $d(y, x) > 0$ and for any $z \in X$ collinearity of (y, x, z) in (X, d) implies that $x = z$. We shall say that y* **witnesses** *that x is an endpoint.*
(c) An element $x \in X$ is called a **startpoint** *of (X, d) if it is an endpoint of (X, d^{-1}).*

Endpoints in partially ordered sets

Let $(X, \leq)$ be a partially ordered set and $y \in X$. We set $\uparrow \mathbf{y} := \{x \in X : y \leq x\}$ and $\downarrow \mathbf{y} := \{x \in X : y \geq x\}$.

Lemma

Let $(X, \leq)$ be a partially ordered set, d its natural T_0-quasi-metric and $x, y \in X$.
Then x is a startpoint of (X, d) witnessed by y if and only if x is a minimal element in $X \setminus \downarrow y$.
Dually, x is an endpoint of (X, d) witnessed by y if and only if x is a maximal element in $X \setminus \uparrow y$.

Another example

Example

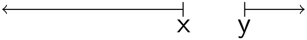
Let X be a set having at least two points and equipped with the discrete order $=$. Then the natural T_0-quasi-metric d of $=$ on X is the discrete metric. Each point of X is an endpoint and a startpoint in (X, d), witnessed by any other point.

Linearly ordered sets

Let $(X, \leq)$ be a linearly ordered set and let $a, b \in X$ be such that $a < b$, but that there does not exist an element $z \in X$ such that $a < z < b$. The pair (a, b) is called a **jump** in X.

Proposition

Let $(X, \leq)$ be a linearly ordered set equipped with its natural T_0-quasi-metric d. The first elements of jumps in X are exactly the endpoints of (X, d). The second elements of jumps in X are exactly the startpoints of (X, d).



Further examples

Example

For a set X with at least one element consider the complete lattice $(\mathcal{P}(X), \subseteq)$ equipped with its natural T_0-quasi-metric d where $\mathcal{P}(X)$ is the powerset of X. Then the startpoints of $(\mathcal{P}(X), d)$ are exactly the singletons. The endpoints of $(\mathcal{P}(X), d)$ are exactly the complements of the singletons.

Example

Let $\mathcal{R}$ be the usual topology on the set $\mathbb{R}$ of the reals equipped with set-theoretic inclusion as a partial order and let d be its natural T_0-quasi-metric. Then there are no startpoints and exactly the complements of singletons are the endpoints in $(\mathcal{R}, d)$.

Completely join-irreducible elements

Definition

An element x in a complete lattice X is called **completely join-irreducible** *if for each subset S of X, $x = \bigvee S$ implies that $x \in S$.*

Completely meet-irreducible elements are defined dually.

Endpoints in complete lattices

Corollary

Let X be a complete lattice and d its natural T_0-quasi-metric. Then $x \in X$ is a startpoint in (X, d) if and only if x is completely join-irreducible.
Similarly, $x \in X$ is an endpoint in (X, d) if and only if x is completely meet-irreducible in (X, d).

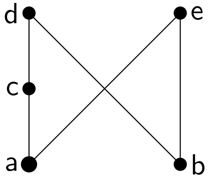


Figure: Hasse Diagram of P_4

b is maximal in $X \setminus \uparrow a$.
c is maximal in $X \setminus \uparrow b$.
d is maximal in $X \setminus \uparrow e$.
e is maximal in $X \setminus \uparrow d$.

In P_4 the set of startpoints is $\{a, b, c, e\}$ and the set of endpoints is $\{b, c, d, e\}$. In particular b is an endpoint, although b is a maximal lower bound of $\{d, e\}$.

Join-density

A subset E of a partially ordered set X is called **join-dense** in X provided that for each $x \in X$ there exists $E' \subseteq E$ such that $x = \bigvee E'$.
Dually one defines the concept of a **meet-dense** subset of a partially ordered set X.

Application of join-density

Proposition

Let X be a partially ordered set and d its natural T_0-quasi-metric.
(a) If E is a join-dense subset of X, then all startpoints of (X, d) belong to E. Dually, if E is a meet-dense subset in X, then all endpoints of (X, d) belong to E.
(b) If E is join- and meet-dense in X, then all startpoints (resp. endpoints) of X are startpoints (resp. endpoints) of E.

Some existence theorem

A T_0-quasi-metric space (X, d) is called **joincompact** provided that $\tau(d^s)$ is compact.

Proposition

Let (X, d) be a joincompact T_0-quasi-metric space with $y_1, y_2 \in X$ such that $d(y_1, y_2) > 0$. There exist a startpoint s in (X, d) and an endpoint e in (X, d) such that (s, y_1, y_2, e) is collinear in (X, d).

Joincompactness continued

Proposition

Let (X, d) be a joincompact T_0-quasi-metric space. Then (Q_X, D) is joincompact and has exactly the same endpoints and startpoints as (X, d).
The injective hull of a joincompact T_0-quasi-metric space X can be identified with the injective hull of the T_0-quasi-metric subspace B of X which consists of all the startpoints and endpoints of X.

Injective partially ordered sets

(1967: B. Banaschewski and G. Bruns) A partially ordered set is injective if and only if it is a complete lattice. (Here we use monotonically increasing maps as morphisms.)
A map $f : (X, d) \to (\{0, 1\}, u)$ is nonexpansive if and only if f is monotonically increasing. (Of course, here u also denotes the restriction of u to $\{0, 1\}^2$.)

The Dedekind-MacNeille completion

Let $(X, \leq)$ be a partially ordered set and let $A \subseteq X$. Then we define the **set of upper bounds** of A, that is, $A^u = \{x \in X : a \leq x \text{ whenever } a \in A\}$ and the **set of lower bounds** of A, that is, $A^\ell = \{x \in X : a \geq x \text{ whenever } a \in A\}$. Let $\mathbf{DM(X)} = \{A \subseteq X : A^{u\ell} = A\}$. The partially ordered set $(DM(X), \subseteq)$ is a complete lattice, known as the **Dedekind-MacNeille completion of** X.

Furthermore $\phi : X \to DM(X)$ defined by $\phi(x) = \downarrow x$ is an order-embedding such that $\phi(X)$ is both join-dense and meet-dense in $DM(X)$.

This is indeed the characteristic property of the Dedekind-MacNeille completion.

Firmness of endpoints and startpoints

Proposition

Let $(X, \leq)$ be a partially ordered set and d its natural T_0-quasi-metric. Furthermore let D be the natural T_0-quasi-metric of $(DM(X), \subseteq)$. Then (X, d) and $(DM(X), D)$ have the same startpoints (resp. endpoints).

Example P_4 continued

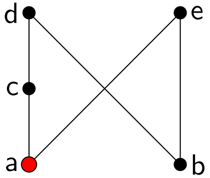


Figure: Hasse Diagram of P_4: a is not an endpoint

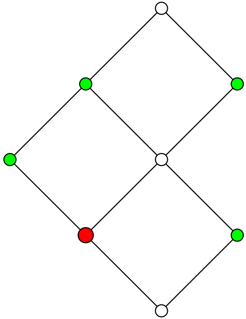


Figure: Hasse Diagram of $DM(P_4)$: a not meet-irreducible, b meet-irreducible

Conclusion

Considering P_4 as a subset of $DM(P_4)$, in the complete lattice $DM(P_4)$ the set of the startpoints of P_4 becomes the set of the (completely) join-irreducible elements of $DM(P_4)$ and the set of the endpoints of P_4 becomes the set of the (completely) meet-irreducible elements of $DM(P_4)$.

Completeness versus q-hyperconvexity

Example

Let $X = \{0, 1\}$ be equipped with its usual order $\leq$ and with its natural T_0-quasi-metric d. Then (Q_X, D) can be identified with $([0, 1], u)$ under the obvious inclusion $X \to [0, 1]$. Hence (X, d) is not q-hyperconvex, although $(X, \leq)$ is a complete lattice.

Proposition

Let (X, d) be a bounded q-hyperconvex T_0-quasi-metric space and $\leq$ its specialization order. Then $(X, \leq)$ is a complete lattice.

q-hyperconvex hull as an extension of Dedekind-MacNeille completion

Let $(X, \leq)$ be a partially ordered set and d its natural T_0-quasi-metric. Furthermore let F_X be the set of all those minimal ample function pairs (f_1, f_2) on (X, d) that attain only the values 0 and 1.

Lemma

In this situation consider an arbitrary pair (f_1, f_2) *of functions* $X \to \{0,1\}$. *Then the following conditions are equivalent:*
(a) $(f_1, f_2) \in F_X$.
(b)

$$f_1(x) = \sup\{d(y,x) \dot{-} f_2(y) : y \in X\}$$

and

$$f_2(x) = \sup\{d(x,y) \dot{-} f_1(y) : y \in X\}$$

whenever $x \in X$.
(c) $f_1^{-1}\{0\} = (f_2^{-1}\{0\})^u$ *and* $f_2^{-1}\{0\} = (f_1^{-1}\{0\})^\ell$.
(d) $(f_2^{-1}\{0\})^{u\ell} = f_2^{-1}\{0\}$ *and* $f_1(x) = \sup_{y \in X}(d(y,x) \dot{-} f_2(y))$ *whenever* $x \in X$.

Embedding $DM(X)$ into (Q_X, D)

Proposition

Let $(X, \leq)$ be a partially ordered set with its natural T_0-quasi-metric d and let F_X be the set of all those minimal ample function pairs (f_1, f_2) on (X, d) that only attain the values 0 and 1.

Then the map $\psi : (F_X, \leq_D) \to (DM(X), \subseteq)$ defined by $(f_1, f_2) \mapsto f_2^{-1}\{0\}$ is an order-isomorphism between F_X (equipped with the specialization order $\leq_D$ induced on F_X by the T_0-quasi-metric D of the q-hyperconvex hull of (X, d)) and the Dedekind-MacNeille completion $(DM(X), \subseteq)$ of X. Furthermore for each $x \in X$, $\psi(f_x) = \downarrow x$.

Characterization of $DM(X)$ as a subspace of Q_X

Remark

Given a partially ordered set $(X, \leq)$ equipped with its natural T_0-quasi-metric d and its q-hyperconvex hull $Q_{(X,d)}$, the subspace S identified above with $DM(X)$ in $Q_{(X,d)}$ is characterized by the property that it is the largest subspace of $Q_{(X,d)}$ containing $e(X)$ such that the T_0-quasi-metric D restricted to $S \times S$ attains only values in $\{0, 1\}$.

Final example

Example

Let $X = \{0, 1\}$ *be equipped with the discrete order* $=$.
The natural T_0*-quasi-metric on* X *is the discrete metric.*
Furthermore (Q_X, D) *can be identified with the set*
$Y = [0, 1] \times [0, 1]$ *equipped with the* T_0*-quasi-metric*

$$D((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = (\alpha_1 \dot{-} \beta_1) \vee (\alpha_2 \dot{-} \beta_2)$$

whenever $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in Y$,
where 0 *is identified with* $(0, 1)$ *and* 1 *is identified with* $(1, 0)$.

Final example continued I

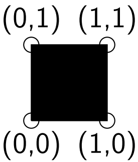
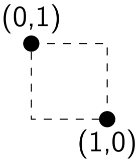


Figure: Unit square equipped with the maximum T_0-quasi-metric; it is Q_X for the subspace X given below



Final example continued II

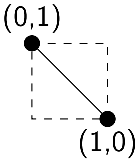


Figure: M_X as subspace of Q_X is isometric to the real unit interval

Final example continued III

The Dedekind-MacNeille completion of (X, d) consists only of the four corner points of $Y = Q_X$ endowed with the induced specialization order on Y.

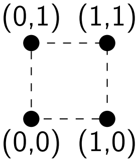


Figure: $DM(X, =)$; drawn by its Hasse Diagram (orientation not according to usual convention: $(0, 0)$ is bottom and $(1, 1)$ is top)

Summary

Proposition

(Isbell) A compact injective metric space Y has a smallest closed subset B such that the hyperconvex hull of B is equal to Y.

Proposition

(Davey and Priestley) A lattice L **with no infinite chains** *is order-isomorphic to the Dedekind-MacNeille completion of the partially ordered set $\mathcal{J}(L) \cup \mathcal{M}(L)$, where $\mathcal{J}(L)$ denotes the set of (completely) join-irreducible elements of L and $\mathcal{M}(L)$ denotes the set of (completely) meet-irreducible elements of L.*
Furthermore $\mathcal{J}(L) \cup \mathcal{M}(L)$ is the smallest subset of L which is both join- and meet-dense in L.

Some references

N. Aronszajn and P. Panitchpakdi, Extension of uniformly continuous transformations and hyperconvex metric spaces, Pacific J. Math. 6 (1956), 405–439.

J.R. Isbell, Six theorems about injective metric spaces, Comment. Math. Helvetici 39 (1964), 65–76.

A.W.M. Dress, Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces, Adv. Math. 53 (1984), 321–402.

R. Espínola and M.A. Khamsi, Introduction to hyperconvex spaces, in: Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001, pp. 391–435.

The q-hyperconvex hull

H. Hirai and S. Koichi, On tight spans for directed distances, Ann. Comb. 16 (2012), 543–569.

E. Kemajou, H.-P.A. Künzi and O.O. Otafudu, The Isbell-hull of a di-space, Topology Applications 159 (2012), 2463–2475.

G. Gutierres and D. Hofmann, Approaching metric domains, Appl. Categor. Struct. 21 (2013), 617–650.

S. Willerton, Tight spans, Isbell completions and semi-tropical modules, Theory and Applications of Categories, Vol. 28, No. 22, 2013, pp. 696–732.

More on orders and categories

B.A. Davey and H.A. Priestley, Introduction to Lattices and Order, Cambridge University Press, 2002.

F.W. Lawvere, Metric spaces, generalized logic, and closed categories, Reprints in Theory and Applications of Categories 1 (2002), 1–37.

E.M. Jawhari, M. Pouzet, and D. Misane, Retracts: graphs and ordered sets from the metric point of view, in: Combinatorics and Ordered Sets, Contemp. Math. 57 (1986), pp. 175–226.

THANK YOU!