The Bishop-Phelps-Bollobás property

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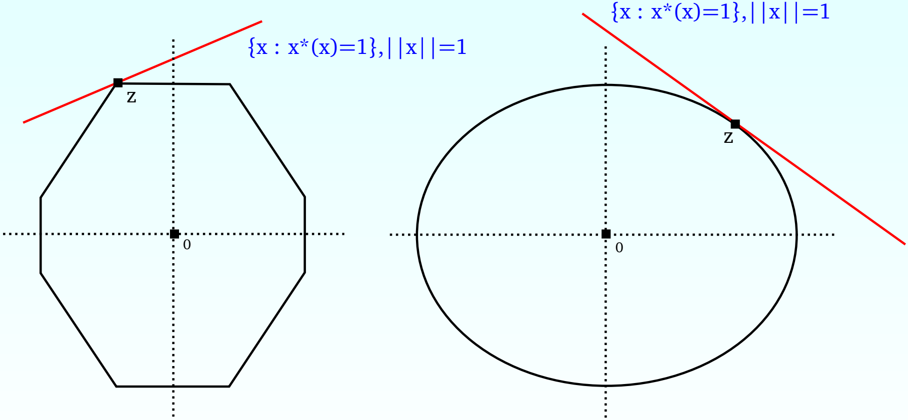
Consider finite dimensional space X.



If a Banach space is finite dimensional, then every functionals attains its norm.

Fact

If a Banach space is reflexive, then every functionals attains its norm.

For arbitrary Banach space? No!

Let

$$x^* = \left(\frac{1}{2^i}\right)_{i=1}^{\infty} \in \ell_1 (= {c_0}^*).$$

For every $x = (x_i)_{i=1}^{\infty} \in B_{c_0}$,

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Example Let us consider

$$x^* = (a_1, a_2, a_3, ..., a_n, ...) \in c_0^* = \ell_1.$$

Then, for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ so that

$$\sum_{i>N} |a_i| < \epsilon$$

.

Set

$$y^* = (a_1, a_2, a_3, ..., a_N, 0, 0, 0...). \text{ Then, } \|x^* - y^*\| < \epsilon.$$

This functional attains its norm at

$$(sign(a_1), sign(a_2), sign(a_3), ..., sign(a_n), 0, 0, 0...) \in c_0$$

This implies that the set of norm attaining functionals is dense in ℓ_1.

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Theorem

E. Bishop, R.R. Phelps(1961) For every Banach space X, the set of norm attaining functionals is dense in its dual space X^*.

1. J. Lindenstrauss (1963) For every Banach space Y, if X is reflexive, then the set of norm-attaining operators is dense in $\mathcal{L}(X,Y)$.
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Back grounds

X, Y : Banach space.

B_X : Closed unit ball of X.

S_X : Closed unit sphere of X.

$\mathscr{L}(X,Y)$: Banach space of all bounded linear operators from X into Y.

Definition

We say that an operator $T \in \mathscr{L}(X,Y)$ *attains its norm* if there exists a point $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\| = \sup\{\|T(x)\| \ : \ x \in B_X\}$.

Definition

We say that a point $x \in S_X$ is a strongly exposed point of B_X if $x^*(x) = 1$ and $(x^*(x_i))_{i=1}^{\infty}$ converges to 1 for some $x^* \in S_{X^*}$ and $(x_i)_{i=1}^{\infty} \subset B_X$, then $(x_i)_{i=1}^{\infty}$ converges to x.

Definition

For every $\epsilon \in (0, 2]$, the *modulus of convexity* of a Banach space $(X, \|\cdot\|)$ is defined by

$$\delta(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| \ : \ x,y \in B_X, \|x-y\| > \epsilon\},$$

and for $\tau > 0$, the *modulus of smoothness* of $(X, \|\cdot\|)$ is defined by

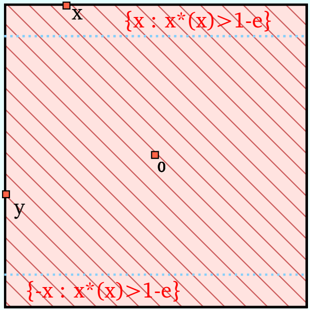
$$\rho(\tau) = \sup\{\frac{\|x+\tau h\| + \|x-\tau h\| - 2}{2} \ : \ \|x\| = \|h\| = 1\}.$$

A Banach space $(X, \|\cdot\|)$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, and uniformly smooth if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$.

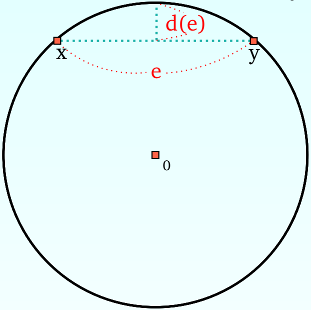
Definition

A Banach space X is said to be *lush* if for every $x, y \in S_X$ and for every $\epsilon > 0$ there is a slice $S = S(B_X, x^*, \epsilon) \subset B_X$, $x^* \in S_{X^*}$, such that $x \in S$ and $dist(y, aconv(S)) < \epsilon$, where $S(B_X, x^*, \epsilon) = \{x \in B_X \ : \ \mathbf{Re}\, x^*(x) > 1 - \epsilon\}$.

Lushness



Uniform Convexity



Following typical spaces are good enough to see the geometric properties of Banach spaces.

Example

Let $p \in [1, \infty)$. The space ℓ_p denotes the vector space of all scalarvalued sequences $x = (x_i)_{i=1}^{\infty}$ satisfying $\sum |x_i|^p < \infty$, endowed with the norm $\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}$. The space ℓ_∞ denotes the vector space of all bounded scalar-valued sequences endowed with the norm $\|x\|_\infty = sup\{|x_i| \ : \ i \in \mathbb{N}\}$. The space c_0 denotes the subspace of ℓ_∞ consisting of all $x = (x_i)$ such that $\lim_{i\to\infty} |x_i| = 0$.

Example

Let $p \in [1, \infty)$. The space $L_p(\mu)$ denotes the vector space of all classes of Lebesgue-measurable scalar functions f defined almost everywhere such that $\int |f|^p d\mu < \infty$, endowed with the norm $\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p}$.

2. Bishop-Phelps-Bollobás Theorem

Theorem

B. Bollobás(1970) *For an arbitrary* $\epsilon > 0$, *if* $x \in B_X$ *and* $x^* \in S_{X^*}$ *satisfy* $|1 - x^*(x)| < \frac{\epsilon^2}{4}$, *then there are* $y \in S_X$ *and* $y^* \in S_{X^*}$ *such that* $y^*(y) = 1$, $\|y - x\| < \epsilon$ *and* $\|y^* - x^*\| < \epsilon$.

Question

1. Can we extend Bishop-Phelps-Bollobás Theorem to operator space between Banach spaces?
2. Can we classify the pair of Banach spaces which satisfies the Bishop-Phelps-Bollobás Theorem for operators?

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2. Can we classify the pair of Banach spaces which satisfies the Bishop-Phelps-Bollobás Theorem for operators?

Definition

M.D. Acosta, R.M. Aron, D. García and M. Maestre (2008) Let X and Y be real or complex Banach spaces. We say that the couple (X,Y) has the Bishop-Phelps-Bollobás property for operators (*BPBP*), if given $\epsilon > 0$ there exist $\beta(\epsilon) > 0$ and $\eta(\epsilon) > 0$ with $\lim_{\epsilon\to 0^+}\beta(\epsilon) = 0$ such that for $T \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1-\eta(\epsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions :

$$\|Su_0\| = 1, \|x_0 - u_0\| < \beta(\epsilon) \text{ and } \|T - S\| < \epsilon$$

1. The couple (X,Y) has the the *BPBP* for finite dimensional Banach spaces X and Y.
2. If Y has property (β), then the couple (X,Y) has the *BPBP* for every Banach space X.

2-1. The Bishop-Phelps-Bollobás property and ℓ_1.

Definition

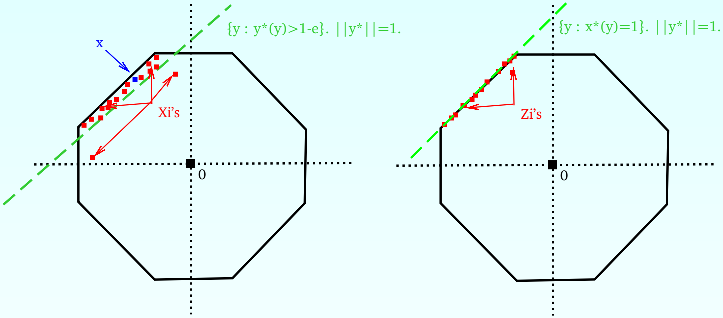
M.D. Acosta, R.M. Aron, D. García and M. Maestre (2008) A Banach space X is said to have the *AHSP* if for every $\epsilon > 0$ there exists $0 < \eta < \epsilon$ such that for every sequence $(x_k) \subset S_X$ and for every convex series $\sum_{n=1}^{\infty} \alpha_k$ with

$$\Big\| \sum_{n=1}^{\infty} \alpha_k x_k \Big\| > 1 - \eta$$

there exist a subset $A \subset \mathbb{N}$ and a subset $\{z_k : k \in A\} \subset S_X$ satisfying

1. $\sum_{k \in A} \alpha_k > 1 - \epsilon$
2.
   1. $\|z_k - x_k\| < \epsilon$ for all $k \in A$
   2. $x^*(z_k) = 1$ for a certain $x^* \in S_{X^*}$ and all $k \in A$

Picture of the AHSP.



Picture 1 : $x = \sum_i \alpha_i x_i$, and $y^*(x) > 1 - \eta$.
Picture 2 : $\|z_i - x_i\| < \epsilon$, and $x^*(z_i) = 1$.

The following Banach spaces have the *AHSP*:

1. a finite dimensional space
2. a real or complex space $L_1(\mu)$ for a σ-finite measure μ
3. a real or complex space $C(K)$ for a compact Hausdorff space K
4. a uniformly convex space

Theorem

M.D. Acosta, R.M. Aron, D. García and M. Maestre (2008) The couple (ℓ_1, Y) has the BPBP if and only if Y has the AHSP.

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Definition

A Banach space X is said to have the alternative Daugavet property (*ADP* for short) if the norm identity

$$\max_{|\omega|=1} \|Id + \omega T\| = 1 + \|T\|$$

holds for every rank-one operators $T \in L(X)$.

Fact

$$Lushness \to (Numerical\ index\ 1) \to ADP$$

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Question Does a Banach space with *ADP* have the *AHSP*? *No!*

Theorem

M. Martín, T. Oikhberg (2004) Let X be a Banach space, K be a compact Hausdorff space, and μ be a positive measure. Then

1. *$C(K,X)$ has the ADP if and only if K is perfect or X has the ADP.*
2. *$L_1(\mu,X)$ has the ADP if and only if μ is atomless or X has the ADP.*

Theorem

Y.S. Choi, S.K. Kim (2011) If $C(K,X)$ has the AHSP, then X has the AHSP.

Let X be a strictly convex Banach space isomorphic to ℓ_1. Then X cannot have the *AHSP*. $C([0.1],X)$ has the *ADP*, but it cannot have the *AHSP*.

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Question Dose a Banach space with both the *ADP* and the *AHSP* have lushness? *No!*

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Y.S. Choi, S.K. Kim (2011) Let (Ω, Σ, μ) be a σ-finite measure space. For a uniformly convex Banach space X, $L_1(\mu, X)$ has the AHSP.

$L_1([0,1], \ell_2)$ has both the *AHSP* and the *ADP*. But, its numerical index is the same as that of ℓ_2, which is smaller than 1. Hence, this is not a lush space.

Question Does a Banach space with numerical index 1 having the *AHSP* have lushness? *Open*

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2-2. The Bishop-Phelps-Bollobás property and $L_1(\mu)$.

We assume in the remainder of this section that μ is a σ-finite measure on an infinite σ-algebra of measurable subsets of Ω.

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Corollary

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Theorem

Y.S. Choi, S.K. Kim (2011) *Let $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$ be families of Banach spaces, $X = \left(\oplus_{i\in I} X_i \right)_{\ell_1}$ and $Y = \left(\oplus_{j\in J} Y_j \right)_{\ell_\infty}$. If the couple (X,Y) has the BPBP, then the couple (X_i, Y_j) also has the BPBP for every $(i,j) \in I \times J$.*

The converse is not true.

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Y.S. Choi, S.K. Kim (2011) *X has the AHSP if and only if $\ell_\infty(X)$ has the AHSP. Equivalently, the couple (ℓ_1, X) has the BPBP if and only if the couple $(\ell_1, \ell_\infty(X))$ has the BPBP.*

Take $X_n = \mathbb{R}$ and $Y_n = Y$, a strictly convex Banach space isomorphic to ℓ_1 for every $n \in \mathbb{N}$. The couple $(\ell_1, \ell_\infty(Y))$ cannot have the *BPBP*, because Y doesn't have the *AHSP*. But the couple $(\mathbb{R}, Y)$ always has the *BPBP*.

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2-3. Bishop-Phelps-Bollobás property and c_0

Study about the *BPBP* on c_0.

1. M.D. Acosta, R.M. Aron, D. García and M. Maestre (2008) For a uniformly convex space Y the couple (ℓ_∞^n, Y) has the *BPBP* for every $n \in \mathbb{N}$.
2. Aron, Cascales and Kozhushkina (2011) Let L be a locally compact Hausdorff space. Then, the couple $(X, C_0(L))$ has the *BPBP* if X is Asplund, and also that if L is scattered, then it is true for all Banach spaces X.

In the result of Acosta et al, the real valued funtions which appear in the definition of *BPBP* depend on the dimension n.

Theorem

S.K. Kim (2011) Given $1 > \epsilon > 0$, assume that Y is a uniformly convex Banach space with its modulus of convexity $0 < \delta(\epsilon) < 1$. There exists a real valued fuction $1 > \delta(\epsilon) > \eta(\epsilon) > 0$ such that if $T \in S_{\mathscr{L}(\ell_\infty^n, Y)}$ and $x \in S_{\ell_\infty^n}$ satisfy

$$\|Tx\| > 1 - \eta(\epsilon)^2$$

, then there exist $S \in S_{\mathscr{L}(\ell_\infty^n, Y)}$ and $x_0 \in S_{\ell_\infty^n}$ such that $\|Sx_0\| = 1$, $\|S - T\| < \epsilon$ and $\|x - x_0\| < \sqrt{\epsilon} + \sqrt{\epsilon^2 + 2\epsilon}$ for every $n \in \mathbb{N}$.

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Lemma

M.D. Acosta, R.M. Aron, D. García and M. Maestre (2008) *Let* $1 > \epsilon > 0$ *be given and* Y *be a uniformly convex Banach space with modulus of covexity* $\delta(\epsilon)$. *If* $T \in S_{\mathcal{L}(c_0,Y)}$, *and* $A \subset \mathbb{N}$ *has the property that* $\left\|TP_A\right\| > 1 - \delta(\epsilon)$, *then we have that* $\left\|T(I - P_A)\right\| \leq \epsilon$, *where* P_A *is the canonical projection from* c_0 *to* ℓ_∞^A.

Corollary

The couple of Banach spaces (c_0, Y) *has the BPBP for uniformly convex* Y.

Theorem

S.K. Kim (2011) *Let* X *be a real Banach space* c_0 *or* ℓ_∞^n $(n \geq 2)$ *and let* Y *be a real strictly convex space. Then* (X, Y) *has the BPBP if and only if* Y *is uniformly convex.*

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1. Let X be a complex Banach space c_0 and Y be a complex strictly convex space. Then (X, Y) has the *BPBP* if and only if Y is uniformly convex?
2. Can we characterize a Banach space Y such that (c_0, Y) has the *BPBP*?

2-4. Bishop-Phelps-Bollobás property and uniformly convex spaces

1. James A Banach space is reflexive if and only if its every bounded linear functional attains its norm.
2. Bourgain A Banach space X has the *Radon-Nikodým property* if and only if for every nonempty closed bounded convex subset C of X and for every Banach space Y, the set of all bounded linear operators T such that $\|T(\cdot)\|$ attains its maximum on C is dense in $L(X, Y)$.

Theorem

S.K. Kim, H.J. Lee (2011) A Banach space X is uniformly convex if and only if for all $\epsilon > 0$ there is some $\eta(\epsilon) > 0$ such that for all $f \in S_{X^*}$ and $x \in B_X$ satisfying $|f(x)| > 1 - \eta(\epsilon)$ there exists some $x_0 \in S_X$ satisfying $|f(x_0)| = 1$, $\|x - x_0\| < \epsilon$.

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Theorem

S.K. Kim, H.J. Lee (2011) *For a uniformly convex Banach space X with its modulus of convexity $\delta(\epsilon) > 0$, (X,Y) has the BPBP for every Banach space Y. More precisely, given $0 < \epsilon < 1$ for any Banach space Y if both $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy*

$$\|Tx\| > 1 - \frac{\epsilon}{2^5}\delta(\frac{\epsilon}{2}),$$

then there exist $S \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ such that $\|Sx_0\| = 1$, $\|S-T\| < \epsilon$ and $\|x - x_0\| < \epsilon$.

The real valued functions are not depend on target spaces.

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The real valued functions are not depend on target spaces.

Theorem

S.K. Kim, H.J. Lee (2011) *Let X be a Banach space. Suppose that given $\epsilon > 0$ there exist positive real valued functions $\eta(\epsilon)$ and $\beta(\epsilon)$ which go to 0 as ϵ goes to 0 satisfying the following :*
For every Banach space Y if $\|Tx\| > 1 - \eta(\epsilon)$ for $T \in S_{\mathscr{L}(X,Y)}$ and $x \in S_X$, there exist $u \in S_X$ and $S \in S_{\mathscr{L}(X,Y)}$ such that
$\|Su\| = 1, \|x - u\| < \beta(\epsilon)$ *and* $\|T - S\| < \epsilon$.

1. *Then, there is no face of S_X which contains a relatively open subset of S_X when X is a real Banach space.*
2. *If X is isomorphic to a strictly convex Banach space, then the set of all extreme points of B_X is dense in S_X.*
3. *If X is isomorphic to a uniformly convex Banach space, then the set of all strongly exposed points of B_X is dense in S_X.*

3. Bishop-Phelps-Bollobás Theorem for non-linear mappings

Definition

Let X and Y be Banach spaces over $\mathbb{K}$. We say that $\mathcal{L}(^nX_1 \times \cdots \times X_n, Y)$ has the Bishop-Phelps-Bollobás property for n-linear mappings (in short *BPBP*), for given $\epsilon > 0$ there exist $\beta(\epsilon) > 0$ and $\eta(\epsilon) > 0$ with $\lim_{\epsilon \to 0^+} \beta(\epsilon) = 0$ such that if there exist both $T \in S_{\mathcal{L}(^nX_1 \times \cdots \times X_n, Y)}$ and $(x_1, ..., x_n) \in S_{X_1} \times \cdots \times S_{X_n}$ satisfying $\|T(x_1, ..., x_n)\| > 1 - \eta(\epsilon)$, then there exist both an n-linear mappings $S \in S_{\mathcal{L}(^nX_1 \times \cdots \times X_n, Y)}$ and $(u_1, ..., u_n) \in S_{X_1} \times \cdots \times S_{X_n}$ such that

$$\|S(u_1, ..., u_n)\| = 1, \|x_i - u_i\| < \beta(\epsilon) \text{ for every } i \in \{1, ...n\} \text{ and } \|T - S\| < \epsilon.$$

Definition

We say that the couple $\mathscr{P}(^nX, Y)$ has the Bishop-Phelps-Bollobás property for n-homogeneous polynomials(in short *BPBP*), for given $\epsilon > 0$ there exist $\beta(\epsilon) > 0$ and $\eta(\epsilon) > 0$ with $\lim_{\epsilon \to 0^+} \beta(\epsilon) = 0$ such that if there exist both $P \in S_{\mathscr{P}(^nX,Y)}$ and $x_0 \in S_X$ satisfying $\|Px_0\| > 1 - \eta(\epsilon)$, then there exist both $S \in S_{\mathscr{P}(^nX,Y)}$ and $u_0 \in S_X$ such that

$$\|Su_0\| = 1, \|x_0 - u_0\| < \beta(\epsilon) \text{ and } \|P - S\| < \epsilon.$$

Theorem

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Definition

M.D. Acosta, J.B. Guerrero, D. García, M. Maestre (2011) For a Banach space X, we will say that the pair (X, X^*) have the *AHSP* if for every $\epsilon > 0$ there exists $0 < \delta, \eta < \epsilon$ such that for every convex series $\sum_{n=1}^{\infty} \alpha_k$ and for every sequence $(x_k^*) \subset S_{X^*}$ and $x_0 \in S_X$ with

$$\sum_{n=1}^{\infty} \alpha_k x_k^*(x_0) > 1 - \eta$$

there exist a subset $A \subset \mathbb{N}$ and a subset $\{z_k : k \in A\} \subset S_X$ and $z_0 \in S_X$ satisfying

1. $\sum_{k \in A} \alpha_k > 1 - \delta$
2.
   1. $\|z_k^* - x_k^*\| < \epsilon$ and $\|z_0 - x_0\| < \epsilon$ for all $k \in A$
   2. $z_k^*(z_0) = 1$ for all $k \in A$

The pair (X,X^*) has the *AHSP* in the following cases:
(1) X is finite dimensional.
(2) X is uniformly smooth.
(3) X is $C_0(\Omega)$, where Ω is any Hausdorff and locally compact topological space (either real or complex case).
(4) X is $K(H)$ space, the space of compact operators on a Hilbert space H.

Theorem

M.D. Acosta, J.B. Guerrero, D. García, M. Maestre (2011) $\mathcal{L}(^2\ell_1 \times X)$ *has the BPBP for bilinear forms if and only if the pair* (X,X^*) *have the AHSP.*

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If the couple $(L_1(\mu),Y)$ *has the BPBP for bilinear forms, then* (Y,Y^*) *has the AHSP.*

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Corollary

1. *Assume that X is a finite dimensional Banach space. Then, $(L_1(\mu), X)$ has the BPBP for bilinear forms . In particular, $(X, L_\infty(\mu))$ has the BPBP for operators.*
2. *$(L_1(\mu), c_0)$ has the BPBP for bilinear forms. In particular, $(c_0, L_\infty(\mu))$ has the BPBP for operators.*
3. *$(L_1(\mu), L_1(m))$ cannot have the BPBP for bilinear forms for any infinite dimensional $L_1(m)$ with a sigma-finite measure m.*
4. *Assume that X is a smooth Banach space. Then, $(L_1(\mu), X)$ has the BPBP for bilinear forms if and only if X is uniformly smooth.*

Theorem

Assume that X, X_i for every $i \in 1, \ldots, n$ are uniformly convex with modulus of convexity $0 < \delta(\epsilon) < 1$. Then for every Banach space Y,

1. $\mathcal{L}({}^nX_1 \times \cdots \times X_n, Y)$ *has the BPBP.*
2. $\mathcal{P}({}^nX, Y)$ *has the BPBP.*

Theorem

When Y has the property (β),

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Thank you for listening!

presented by Sun Kwang Kim.