COMPLETENESS TYPE PROPERTIES AND SPACES OF CONTINUOUS FUNCTIONS

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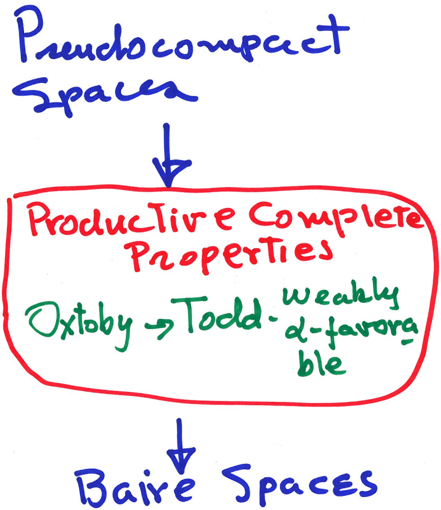
Initial conditions

In this talk *space* will mean *Tychonoff space with more than one point*.

Introduction

- Pseudocompact and Baire spaces are outstanding classes of spaces, but these properties are not productive.

- Efforts have been made to define classes of spaces which contain all pseudocompact spaces, satisfy the Baire Category Theorem and are closed under arbitrary topological products.


Pseudocompact Spaces
Productive Complete Properties
Oxtoby → Todd - weakly α-favorable
Baire Spaces

- One of these properties is Oxtoby completeness.
- Another is Todd completeness.

Definition 2.1.

A family $\mathcal{B}$ of sets in a topological space X is called π*-base*

(respectively, π*-pseudobase*)

if every element of $\mathcal{B}$ is open

(respectively, has a nonempty interior)

and every nonempty open set in X contains an element of $\mathcal{B}$.


∀ A∈J
A
B
∃ B∈B⊆J
(X, J)
π-base B

Definition 2.2.

A space X is *Oxtoby complete* (respectively, *Todd complete*) if there is a sequence

$$\{\mathcal{B}_n : n < \omega\}$$

of π-bases, (respectively, π-pseudobases) in X such that for any sequence $\{U_n : n < \omega\}$ where

$$U_n \in \mathcal{B}_n \text{ and } \mathrm{cl}_X U_{n+1} \subseteq \mathrm{int}_X U_n \text{ for all } n,$$

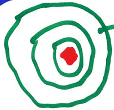
then

$$\bigcap_{n<\omega} U_n \neq \emptyset.$$

Oxtoby Sequence

$\{\mathcal{B}_n : n<\omega\}$

$\mathcal{B}_n$ is π-base



$B_n \in \mathcal{B}_n$

$cl\, B_{n+1} \subseteq B_n$

$\Rightarrow x \in \bigcap_n B_n \neq \emptyset$

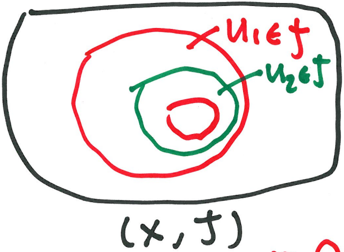
(X, τ)

There are also, properties of type completeness defined by topological games:

Definition 2.3

A space Z is *weakly α-favorable* if Player II has a winning strategy in the Banach-Mazur game $\mathrm{BM}(Z)$.

Banah-Mazur Game



Player I wins if $\bigcap_n U_n = \emptyset$

Player II wins if $\bigcap_n U_n \neq \emptyset$

Completeness type properties

The relations between all these properties are:

Pseudocompact $\Rightarrow$ Oxtoby complete $\Rightarrow$ Todd complete

Todd complete $\Rightarrow$ weakly α-favorable $\Rightarrow$ Baire

We know:

Proposition 3.1

$C_p(X)$ is never pseudocompact.

Proposition 3.2, van Douwen, Pytkeev

$C_p(X)$ is a Baire space iff every pairwise disjoint sequence of finite subsets of X has a strongly discrete subsequence.

- We want to say something about the completenes properties just presented in spaces of the real-valued continuous functions with the pointwise convergence topology $C_p(X)$. Mainly, we want to relate these properties with properties defined in X. In order to do this we define:

Definition 3.3

A space X is *u-discrete* if every countable subset of X is discrete and C-embedded in X.

D.J. Lutzer and R.A. McCoy analyzed Oxtoby pseudocompleteness in $C_p(X)$. They proved:

Theorem 3.2, 1980

Let X be a pseudonormal space. Then the following are equivalent:
1.- X is u-discrete.

2.- $C_p(X)$ is Oxtoby complete.

3.- $C_p(X)$ is weakly α-favorable.

4.- $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.

Afterwards, A. Dorantes-Aldama, R. Rojas-Hernández and Á. Tamariz-Mascarúa improve the Lutzer and McCoy result:

Theorem 3.3, 2015

Let X be a space with property D of van Douwen. Then the following are equivalent:

1.-X is u-discrete.

2.- $C_p(X)$ is Todd complete.

3.- $C_p(X)$ is Oxtoby complete.

4.- $C_p(X)$ is weakly α-favorable.

5.- $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.

And A. Dorantes-Aldama and D. Shakhmatov proved:

Theorem 3.4, 2016

The following staments are equivalent:

1.- X is u-discrete.

2.- $C_p(X)$ is Todd complete.

3.- $C_p(X)$ is Oxtoby complete.

4.- $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.

Finally, S. García-Ferreira, R. Rojas-Hernández and Á. Tamariz-Mascarúa proved:

Theorem 3.5, 2016

$C_p(X)$ is weakly α-favorable if and only if X is u-discrete.

And so,

Theorem 3.6, 2016

The following conditions are equivalent.

1.- X is u-discrete;

2.- $C_p(X)$ is Todd complete;

3.- $C_p(X)$ is Oxtoby complete;

4.- $C_p(X)$ is weakly α-favorable;

5.- $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.

Weakly pseudocompact spaces

Theorem 4.1. (Hewitt, 1948)

A space X is pseudocompact if and only if it is G_δ-dense in βX (iff it is G_δ-dense in any of its compactifications).

- So, a natural generalization of pseudocompactness is:

Definition 4.2. (García-Ferreira and García-Máynez, 1994)

A space is *weakly pseudocompact* if it is G_δ-dense in some of its compactifications.

- Then every pseudocompact space is weakly pseudocompact.

Weakly pseudocompact spaces

Theorem 4.3. (García-Ferreira and García-Máynez, 1994)

- Every weakly pseudocompact space is Baire.
- Weak pseudocompactness is productive.

Weakly pseudocompact spaces

- Example of weakly pseudocompact spaces:

  1.- The non-countable discrete spaces.

  2.- (F.W. Eckertson, 1996)
  The metrizable hedgehog $J(\kappa)$ with $\kappa > \omega$.

We can add "$C_p(X)$ is weakly pseudocompact"to the list of the following theorem?

Theorem 3.6, 2016

The following conditions are equivalent.

1.- X is u-discrete;

2.- $C_p(X)$ is Todd complete;

3.- $C_p(X)$ is Oxtoby complete;

4.- $C_p(X)$ is weakly α-favorable;

5.- $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.

With respect to this problem we have:

Theorem 4.3, S. García-Ferreira, R. Rojas-Hernández, Á. Tamariz-Mascarúa, 2016

If $\mathbb{R}^\kappa$ is weakly pseudocompact for some cardinal κ, then $\mathbb{R}^\mathfrak{c}$ is weakly pseudocompact.

Theorem 4.4

The space $\mathbb{R}^\kappa$ is weakly pseudocompact if and only if $C_p(X)$ is weakly pseudocompact for every u-discrete space X with $min\{\mathfrak{c}, \kappa\} \leq |X|$.

Theorem 3.6, 2016

If $\mathbb{R}^{\omega_1}$ is weakly pseudocompact, then the following conditions are equivalent.

1.- X is u-discrete;

2.- $C_p(X)$ is weakly pseudocompact;

3.- $C_p(X)$ is Todd complete;

4.- $C_p(X)$ is Oxtoby complete;

5.- $C_p(X)$ is weakly α-favorable;

6.- $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.

Weakly pseudocompact spaces

Problem 6.1

Is $\mathbb{R}^{\omega_1}$ *weakly pseudocompact*?

One interesting consequence of the results mentioned in this talk that we got are generalizations of the classic Tkachuk's Theorem:

Theorem 5.1, V. Tkachuk, 1987

$C_p(X) \cong \mathbb{R}^\kappa$ if and only if X is discrete of cardinality κ.

Topological groups

We obtained:

Theorem 5.2

Let G be a separable completely metrizable topological group and X a set. If H is a dense subgroup of G^X and H is homeomorphic to G^Y for some set Y, then $H = G^X$.

Corollary 5.3

Let X be a space and let G be a separable completely metrizable topological group. If $C_p(X, G)$ is homeomorphic to G^Y for some set Y, then $C_p(X, G) = G^X$.