Spectra of weighted (LB)-algebras of entire functions on Banach spaces

FNRS-functional analysis meeting

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Outline

1. Weighted inductive limits of spaces of entire functions
2. Weighted algebras of holomorphic functions
3. Algebra homomorphisms between the algebras $A_p(X)$
4. The spectrum of $VH(X)$

k-homogeneous polynomials

Let X be a complex Banach space.

Definition

A mapping $P : X \to \mathbb{C}$ is said to be a k-homogeneous polynomial if there exists a continuous k-linear mapping

$$A : \underbrace{X \times \cdots \times X}_{k)} \to \mathbb{C}$$

such that $P(x) = A(x, ..., x)$ for every $x \in X$. The set of all k-homogeneous polynomial is denoted by $\mathcal{P}(^k X)$.

$(\mathcal{P}(^k X), \| \ \|)$ is a Banach space under the norm:

$$\|P\| = \sup\{|P(x)| : x \in X, \|x\| \leq 1\}, \ \ P \in \mathcal{P}(^k X)$$

Holomorphic functions of bounded type

In order to holomorphic mappings behave as in the finite dimensional case, we introduce the following spaces:

Definition

Consider the holomorphic functions of bounded type:

$$H_b(X) := \{f \in H(X) \text{ such that } f \text{ is bounded on bounded sets}\}$$

and the locally convex topology τ_b of uniform convergence on the bounded subsets of X.

- $(H_b(X), \tau_b)$ is a Fréchet space.
- The Taylor series of each $f = \sum_{k=0}^{\infty} P_k f \in H_b(X)$ at zero converges uniformly on any ball around $x = 0$.

The space $H_v(X)$

A map $v : X \to]0,1]$ is called a weight if there exists a continuous decreasing function $\eta : [0,\infty[\to]0,1]$ with $\lim_{r\to\infty} \eta(r) r^k = 0$ for all $k \in \mathbb{N}$ such that $v(x) = \eta(\|x\|)$ for all $x \in X$.

Definition

Given v a weight on X, we define the weighted Banach spaces of holomorphic functions :

$$H_v(X) := \{f \in H(X) : \|f\|_v := \sup_{x\in X} v(x)|f(x)| < \infty\}$$

Proposition

- *$\mathcal{P}(^k X) \subseteq H_v(X) \subseteq H_b(X)$ for all $k \in \mathbb{N}$.*
- *Given $k \in \mathbb{N}$, the map $P_k : H_v(X) \to H_v(X)$, $f \to P_k f$ is continuous with $\|P_k f\|_v \leq \|f\|_v$ for all $f \in H_v(X)$.*

The inductive limit $VH(X)$

Given $V := \{\upsilon_n\}_{n\in\mathbb{N}}$ a decreasing sequence of weights, i.e., $\upsilon_{n+1}(x) \leq \upsilon_n(x)$ for all $x \in X$, $n \in \mathbb{N}$, observe that $H_{\upsilon_n}(X) \hookrightarrow H_{\upsilon_{n+1}}(X)$ with continuous inclusion.

Definition

We denote by $VH(X)$ the weighted inductive limit of spaces of entire functions:

$$VH(X) := ind_{n\in\mathbb{N}} H_{\upsilon_n}(X)$$

endowed with the inductive limit topology τ, that is, the finest locally convex topology such that the inclusion $H_{\upsilon_n}(X) \hookrightarrow VH(X)$ is continuous for all $n \in \mathbb{N}$.

$(VH(X), \tau)$ is a locally convex Hausdorff space such that $VH(X) \hookrightarrow H_b(X)$ continuously.

Schauder decompositions

A sequence $\{E_n\}_{n\in\mathbb{N}}$ of subspaces of a locally convex space E is a Schauder decomposition of E if:

- $\forall x \in E,\ \exists! \{x_n\}_{n\in\mathbb{N}}, x_n \in E_n,\ x = \sum_{n=1}^{\infty} x_n := \lim_{m\to\infty} \sum_{n=1}^{m} x_n.$
- The projections $\{u_n\}_{n\in\mathbb{N}},\ u_m(\sum_{n=1}^{\infty} x_n) := \sum_{n=1}^{m} x_n$ are continuous.

Proposition

Let $V = \{\upsilon_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of weights. If for each $m \in \mathbb{N}$ there exist $R > 1, D > 0,\ n \in \mathbb{N}, n \geq m$ such that

$$\clubsuit \quad \upsilon_n(x) \leq D\upsilon_m(Rx) \quad \forall x \in X$$

then $\{\mathcal{P}(^k X)\}_{k\geq 0}$ is a Schauder decomposition of $VH(X)$.

Due to Cauchy inequalities we get:

$$\|P_k f\|_{\upsilon_n} \leq D\frac{1}{R^k}\|f\|_{\upsilon_m}\ \forall f \in H_{\upsilon_m}(U),\ k = 0, 1, 2, \ldots$$

Schauder decompositions

Given a weight υ on X, consider the decreasing families of weights $W := \{w_n\}_{n\in\mathbb{N}}$, $w_n(x) = \upsilon(nx)$ and $\vartheta := \{v_n\}_{n\in\mathbb{N}}$, $v_n(x) = \upsilon(x)^n$.

Example

$\{\mathcal{P}(^kX)\}_{k\geq 0}$ is a Schauder decomposition of $WH(X)$.

Example

ϑ satisfies condition $\clubsuit \iff \exists R > 1,\ \alpha, C > 0$ such that $\forall t > 0$:

$$\eta(t)^\alpha \leq C\eta(Rt).$$

In this case, $WH(X) \hookrightarrow \vartheta H(X)$ continuously.

When is the space $VH(X)$ an algebra?

Definition

A locally convex space will be an algebra if it has a locally convex structure with a continuous multiplication.

Theorem

$VH(X)$ is an algebra $\iff \forall n \in \mathbb{N}, \exists m \geq n, C > 0$ *such that* $v_m \leq C\widetilde{v}_n^2$.

Remember that given a weight v we define its associated weight $\widetilde{v}$ by:

$$\frac{1}{\widetilde{v}(z)} := \sup\{|f(z)| : f \in H_v(X),\ \|f\|_v \leq 1\}$$

When is the space $VH(X)$ an algebra?

Given a weight v on X, consider the decreasing families of weights $W := \{w_n\}_{n\in\mathbb{N}}$, $w_n(x) = v(nx)$ and $\vartheta := \{v_n\}_{n\in\mathbb{N}}$, $v_n(x) = v(x)^n$.

Example

The space $\vartheta H(X)$ is always an algebra.

Example

$WH(X)$ is an algebra $\iff$ there exist $k \geq 1,\ C > 0$ such that $\forall t \geq 0$,

$$\eta(kt) \leq C\widetilde{\eta}(t)^2.$$

In this case, if η is essential, $\vartheta H(X) \hookrightarrow WH(X)$ continuously and there exist positive constants a, b and α so that $\eta(t) \leq ae^{-bt^{\alpha}}$ for all $t \geq 0$.

The weight function p

Definition

Let $p : [0, \infty[\to [0, \infty[$ be an increasing continuous function. It is called a weight function if it has the following properties:

1. $\varphi : r \to p(e^r)$ is convex,
2. $log(1 + r^2) = o(p(r))$,
3. $p(2r) = O(p(r))$.

The following functions are easily seen to be weight functions:

- $p(r) = r^d, \; d > 0$
- $p(r) = \begin{cases} (log r)^2 & \text{if } r \geq 1 \\ 0 & \text{if } r \leq 1 \end{cases}$

The space $A_p(X)$

Definition

In what follows, consider the weight $v(x) = \eta(\|x\|) = e^{-p(\|x\|)}$ and the decreasing sequence of weights $\vartheta = \{v^n\}_{n\in\mathbb{N}}$. We denote by

$$A_p(X) := \vartheta H(X)$$

the weighted algebra related to this sequence of weights.

Proposition

$\{\mathcal{P}(^k X)\}_{k\geq 0}$ is a Schauder decomposition of $A_p(X)$.

$$\exists\lambda > 0 \text{ such that } p(2t) \leq \lambda p(t) + \lambda \Longrightarrow \eta(t)^\lambda \leq e^\lambda \eta(2t).$$

Given a convex and increasing function $\varphi : [0,\infty[\to [0,\infty[$ such that $\lim_{t\to\infty} \frac{t}{\varphi(t)} = 0$, the Young-conjugate of φ is defined as

$$\varphi^*(t) := \sup_{x\geq 0}(xt - \varphi(x)).$$

We are going to consider the Young-conjugate of $\varphi(r) = p(e^r)$.

Definition

Given a Banach space X and a weight function p, for each $n \in \mathbb{N}$ consider the Banach space:

$$\ell_{\infty,n}(X,p) := \{(p_j)_j \in \prod_{j\in\mathbb{N}} \mathcal{P}(^jX) : |||(p_j)_j|||_n := \sup_{j\in\mathbb{N}} \|p_j\| \exp\left(n\varphi^*\left(\frac{j}{n}\right)\right) < \infty\}$$

The inclusions $\ell_{\infty,n}(X,p) \hookrightarrow \ell_{\infty,n+1}(X,p)$ are continuous for all $n \in \mathbb{N}$, since

$$\exp\left(n\varphi^*\left(\frac{j}{n}\right)\right) = \sup_{R>0}\{R^j e^{-np(R)}\}.$$

The space $\kappa_\infty(X, p)$

Definition

We denote by $\kappa_\infty(X, p)$ the locally convex space

$$\kappa_\infty(X, p) := ind(\ell_{\infty,n}(X, p), ||| \; |||_n),$$

endowed with the inductive limit topology.

Theorem

Given a weight function p on a Banach space X, the map $\phi : A_p \to \kappa_\infty(X, p)$, $f = \sum_j p_j \mapsto \{p_j\}_j$, where $p_j \in \mathcal{P}(^j X)$, is an algebra topological isomorphism:

$$A_p \cong \kappa_\infty(X, p)$$

The space $\kappa_\infty(X, p)$

Proposition

Let X and Y Banach spaces and suppose that for each $j \in \mathbb{N}$ there exists an isomorphism $\phi_j : \mathcal{P}(^j X) \to \mathcal{P}(^j Y)$, and $a, A, b, B > 0$ such that

$$\|\phi_j(p_j)\| \leq aA^j \|p_j\| \text{ and } \|p_j\| \leq bB^j \|\phi_j(p_j)\| \ \forall p_j \in \mathcal{P}(^j X).$$

Then, the map $\phi : \kappa_\infty(X, p) \to \kappa_\infty(Y, p)$, $\phi(\{p_j\}_j) = \{\phi_j(p_j)\}_j$ is a topological isomorphism for every weight function p.

- Moreover, if

$$\phi_j(p_k q_r) = \phi_k(p_k)\phi_r(q_r) \ \ \forall p_k \in \mathcal{P}(^k X), \ \forall q_r \in \mathcal{P}(^r X), \ r + k = j,$$

$\Longrightarrow \phi$ is an algebra topological isomorphism.

A Banach-Stone type theorem

Consider now the weight function $p(r) = r, \; r \geq 0$. Observe that in this case, $A_p(X) = Exp(X)$.

Lemma

Let $A : Exp(X) \to Exp(Y)$ be an algebra homomorphism. Then Ax^* is a degree 1 polynomial for all $x^* \in X^*$

Theorem

Let X and Y be symmetrically regular spaces or X a regular space.

$$Exp(X) \cong Exp(Y) \iff X^* \cong Y^*.$$

Recall that a complex Banach space X is said to be (symmetrically) regular if every continuous (symmetric) linear mapping $T : X \to X^*$ is weakly compact. T is symmetric if $Tx_1(x_2) = Tx_2(x_1)$ for all $x_1, x_2 \in X$.

The spectrum of $VH(X)$

Let X be a symmetrically regular Banach space and $VH(X)$ an algebra such that V satisfies condition ♣.

Definition

We denote by $V\mathfrak{M}(X)$ the spectrum of $VH(X)$:

$$V\mathfrak{M}(X) = \{\phi \in VH(X)^*,\ \phi \neq 0 \text{ and multiplicative } \}$$

Definition

A Riemann domain spread over a locally convex space E is a pair (Ω, π) consisting of a Hausdorff topological space Ω and a local homeomorphism $\pi : \Omega \to E$.

Known results

- Aron, Cole and Gamelin (1991) $\Longrightarrow$ First steps towards the description of the spectrum $\mathfrak{M}_b(X)$ of $H_b(X)$.
- Aron, Galindo, García and Maestre (1996) $\Longrightarrow$ $\mathfrak{M}_b(U)$ is a Riemann analytic manifold modelled on X^{**}, for U an open subset of X. For $U = X$, $\mathfrak{M}_b(X)$ can be viewed as the disjoint union of analytic copies of X^{**}, this copies being the connected components of $\mathfrak{M}_b(X)$.

The same situation holds for the spectrum $\mathfrak{M}V(U)$ of weighted Fréchet algebras of holomorphic functions $HV(U)$:

- Carando and Sevilla-Peris (2009) $\Longrightarrow$ Spectrum $\mathfrak{M}V(X)$.
- Carando, García, Maestre and Sevilla-Peris (2009) $\Longrightarrow$ Spectrum $\mathfrak{M}V(U)$.

Structure of $V\mathfrak{M}(X)$

Theorem

$V\mathfrak{M}(X)$ is a Riemann analytic manifold modelled on X^{**}. It can be viewed as the disjoint union of analytic copies of X^{**}, this copies being the connected components of $V\mathfrak{M}(X)$.

Sketch of the proof:

$(V\mathfrak{M}(X), \pi)$, where $\pi : V\mathfrak{M}(X) \to X^{**}$, $\pi(\phi) = \phi|_{X^*}$ is a Riemann domain over X^{**} :

- For a fixed $z \in X^{**}$, $\tau_z : VH(X) \to VH(X)$, $(\tau_z f)(x) := \widetilde{f}(J_X x + z)$ is well defined and continuous.
- For a fixed $\phi \in V\mathfrak{M}(X)$ and $\epsilon > 0$,

$$V_{\phi,\epsilon} := \{\phi \circ \tau_z : z \in X^{**},\ \|z\| < \epsilon\} \subseteq V\mathfrak{M}(X)$$

  and $V_\phi := \{V_{\phi,\epsilon}\}_{\epsilon>0}$ forms a neighbourhood basis at ϕ for a Hausdorff topology on $V\mathfrak{M}(X)$.

Structure of $V\mathfrak{M}(X)$

- π maps $V_{\phi,\epsilon}$ homeomorphically onto $B_{X^{**}}(\pi(\phi),\epsilon)$ since $\pi(\phi \circ \tau_z) = \pi(\phi) + z$ for all $\phi \in V\mathfrak{M}(X)$. □

Remark

Since the evaluation mappings belong to the spectrum and the mapping

$$\delta : X^{**} \to V\mathfrak{M}(X),\ \ \delta_z(f) = \widetilde{f}(z)$$

is injective, we can consider X^{**} included in $V\mathfrak{M}(X)$. Observe that $\delta_0 \circ \tau_z = \delta_z$ and $\pi(\delta_z) = z$ for each $z \in X^{**}$ yields the homeomorphism between the connected copy $V_{\delta_0,\infty}$ and X^{**}.

$VH(X) \subseteq H_b(V\mathfrak{M}(X))$

Each $f \in VH(X)$ defines a mapping called the Gel'fand transform:

$$\hat{f} : V\mathfrak{M}(X) \to \mathbb{C},\ \hat{f}(\phi) = \phi(f).$$

Since $\hat{f}(\delta_{x^{**}}) = \widetilde{f}(x^{**})$ for each $x^{**} \in X^{**}$, we can say that each $f \in VH(X)$ can be extended not only to a function $\widetilde{f}$ in $VH(X^{**})$, but also to a function on the spectrum $V\mathfrak{M}(X)$. Moreover, this function is holomorphic:

$VH(X) \subseteq H_b(V\mathfrak{M}(X))$

Theorem

Let X be a symmetrically regular Banach space and V a family of weights satisfying Condition ♣ and such that $VH(X)$ is an algebra. Then, for every $f \in VH(X)$, the Gel'fand transform $\hat{f} : V\mathfrak{M}(X) \to \mathbb{C}$ given by $\hat{f}(\phi) = \phi(f)$ is holomorphic.

Remark:

By the Riemann domain structure of $V\mathfrak{M}(X)$, this means that $\hat{f} \circ (\pi|_{V_{\phi,\infty}})^{-1}$ is holomorphic on X^{**} for all $\phi \in V\mathfrak{M}(X)$, where $V_{\phi,\infty} = \cup_{\varepsilon>0} V_{\phi,\varepsilon}$.

$VH(X) \subseteq VH(V\mathfrak{M}(X))$

Moreover, if

$$\forall m, n \in \mathbb{N},\ n \geq m,\ \exists q \in \mathbb{N},\ \exists \delta > 0 : \sup_{x \in X} \frac{v_n(x)}{v_m(x+z)} \leq \frac{\delta}{v_q(z)}\ \forall z \in X^{**},$$

then we can even get that f belongs to $VH(V\mathfrak{M}(X))$.

Example

This condition is satisfied when $V = \{v^n\}$, $v(x) = \eta(\|x\|)$, and there is $\alpha > 0$ with $\eta(s)\eta(t) \leq \alpha\eta(s+t)$. Ex: $\eta(t) = e^{-t},\ t \geq 0$.

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Spectrum homomorphisms

Every algebra homomorphism $A : Exp(X) \to Exp(Y)$ induces a mapping $\theta_A : V\mathfrak{M}(Y) \to V\mathfrak{M}(X)$ defined by $\theta_A(\phi) = \phi \circ A$.

Theorem

Let X and Y be symmetrically regular Banach spaces and let $A : Exp(X) \to Exp(Y)$ be an algebra homomorphism. TFAE:

(i) *There exist $\phi \in V\mathfrak{M}(X)$ and $T : Y^{**} \to X^{**}$ affine and w^*-w^*-continuous so that $Af(y) = \phi(\widetilde{f}(\cdot + Ty))$ for all $y \in Y$.*

(ii) *θ_A maps sheets into sheets.*

(iii) *θ_A maps Y^{**} into a sheet.*

In particular, θ_A is continuous if and only if it is continuous on Y^{**}.

Spectrum homomorphisms

Definition

The algebra homomorphism $A : Exp(X) \to Exp(Y)$ is an AB-composition homomorphism if there exists an affine mapping $g : Y^{**} \to X^{**}$ such that $\widetilde{Af}(y^{**}) = \widetilde{f}(g(y^{**}))\ \forall f \in Exp(X),\ \forall y^{**} \in Y^{**}$.

Corollary

Let X and Y be symmetrically regular Banach spaces and $A : Exp(X) \to Exp(Y)$ an algebra homomorphism. $\theta_A(Y^{**}) \subseteq X^{**} \Leftrightarrow A$ is an AB-composition homomorphism.