Some generalizations of Burnside’s Theorem

Yangming Li

Guangdong University of Education, Guangzhou

Valencia, 17 Feb., 2014
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Notations

- All groups considered in this talk are finite;
- \( G \) always denotes a finite group;
- \( \pi(G) \) denotes the set of all primes dividing the order of \( G \);
- Suppose that \( p \) is a prime in \( \pi(G) \). \( O^p(G) \) is the subgroup of \( G \) generated by all \( p' \)-elements of \( G \);
- \( Z_k(G) \) is the \( k \)-th center of \( G \), \( k \geq 1 \);
- In fact, \( Z_1(G) = Z(G) \), the center of \( G \), and \( Z_k(G)/Z_{k-1}(G) = Z(G/Z_{k-1}(G)) \) for \( k > 1 \).
If $P$ is a $p$-group and $k$ is a natural number, we denote

$$\Omega_k(P) = \langle x \in P : x^{p^k} = 1 \rangle,$$

$$\Omega(P) = \begin{cases} 
\Omega_1(P), & \text{if } p \text{ is odd}; \\
\Omega_2(P), & \text{if } p = 2.
\end{cases}$$
A group $G$ is called $p$-nilpotent if $G$ has a normal $p$-complement;
i.e., $p$ does not divide the order of $O^p(G)$;
i.e., $P \cap O^p(G) = 1$ for any Sylow $p$-subgroup $P$ of $G$. 
Notations

O. H. Kegel, Math. Z., 1962

A subgroup $H$ of $G$ is said to be $s$-permutable (or $s$-quasinormal, $\pi$-quasinormal) in $G$ if $H$ permutes with every Sylow subgroup of $G$. 
Let $L/K$ be a chief factor of $G$ and $H$ a subgroup of $G$. We say that

i) $H$ covers $L/K$ if $L \leq HK$, i.e., $L/K \leq HK/K$;

ii) $H$ avoids $L/K$ if $L \cap H \leq K$, i.e., $L/K \cap HK/K = 1$;

iii) $H$ has the cover-avoidance property in $G$, $H$ is a CAP-subgroup of $G$ in short, if $H$ either covers or avoids every chief factor of $G$.

The cover-avoidance property of subgroup was first studied by Gaschütz in 1962 to study the solvable groups, later by many other experts.

Let $H$ be a subgroup of a group $G$. Then, we say that $H$ is a partial CAP-subgroup (or semi CAP-subgroup or SCAP-subgroup for some authors) of $G$ if there exists a chief series $\Gamma_H$ of $G$ such that $H$ either covers or avoids each chief factor of $\Gamma_H$. 
For a group $G$, we know that the normalizers of its Sylow subgroups give a lot of messages of the whole group $G$.

M. Bianchi, A. Gillio Berta Mauri and P. Hauck, Arch. Math., 1986

A group is nilpotent if and only if the normalizer of its each Sylow subgroup is nilpotent.
Ballester-Bolinches and Shemetkov strengthen the above result to get:

**Ballester-Bolinches and Shemetkov, Siberian Math. J., 1999**

A group is nilpotent if and only if the normalizer of its each Sylow $p$-subgroup is $p$-nilpotent for any prime $p \in \pi(G)$. 
Background

Here we consider the local version, i.e., referring one prime.

Now we assume that $p$ is a fixed prime in $\pi(G)$ and $P$ is a Sylow $p$-subgroup of $G$. 
The property of $N_G(P)$ still influences the structure of $G$.

**Thompson** Suppose that $p \geq 5$, $P$ is a Sylow $p$-subgroup of $G$ and $P \neq 1$. If $N_G(P)$ is $p$-nilpotent, then $O^p(G) < G$. 
Applying Thompson’s result, it is not difficult to prove a long-standing conjecture of Zassenhaus.

**Theorem** If $G$ is a finite group and $N_G(Q) = Q$ for every Sylow subgroup $Q$ of $G$, then $|G|$ is a power of a prime.
We can see in general:

\[ N_G(P) \text{ is } p\text{-nilpotent} \iff G \text{ is } p\text{-nilpotent}. \]
Example  Suppose that $G = GL(2, 3)$ and $p = 2$ and $P$ is a Sylow 2-subgroup of $G$. Then $N_G(P) = P$ is 2-nilpotent but $G$ is not 2-nilpotent.
Hence, there is the following question:

**Question**  *Suppose that $N_G(P)$ is $p$-nilpotent. Which extra condition can guarantee that $G$ is $p$-nilpotent?*
Background

In the literature, many experts have considered this question, for example, Burnside, P. Hall, Wielandt, Glauberman, Thompson, etc.

**Glauberman-Thompson** Let $p$ be an odd prime divisor of the order of a group $G$ and let $P$ be a Sylow $p$-subgroup of $G$. Then $G$ is $p$-nilpotent if and only if $N_G(Z(J(P)))$ is $p$-nilpotent, where $J(P)$ is the Thompson subgroup of $P$.

**Remark** We note that $N_G(P) \subseteq N_G(Z(J(P)))$. 
Our works are from the most famous Burnside Theorem:

**Burnside Theorem** Let $p$ be a fixed prime in $\pi(G)$ and $P$ a Sylow $p$-subgroup of $G$. Suppose that $N_G(P)$ is $p$-nilpotent. Then $G$ is $p$-nilpotent if $P$ is abelian.

**equivalent form:**

Let $p$ be a fixed prime in $\pi(G)$ and $P$ a Sylow $p$-subgroup of $G$. Then $G$ is $p$-nilpotent if $N_G(P) = C_G(P)$. 
$P$ is abelian $\iff P \leq Z(P)$. From this point of view, to extend Burnside Theorem, we must weaken the condition “$P \leq Z(P)$”.

We have two ways, one is to enlarge “$Z(P)$”, another is to lessen “$P$”.

Let $p$ be a fixed prime in $\pi(G)$ and $P$ a Sylow $p$-subgroup of $G$. Suppose that $N_G(P)$ is $p$-nilpotent. Then $G$ is $p$-nilpotent if the nilpotency class of $P$ is less than $p$.

Remark: the nilpotency class of $P$ is less than $p$ means that $P \leq Z_{p^{-1}}(P)$
Let $G$ be a finite group and $P$ a Sylow $p$-subgroup of $G$. Suppose that $N_G(P)$ is $p$-nilpotent. Then $G$ is $p$-nilpotent if $\Omega_1(P) \leq Z(P)$ and $C_G(Z(P))$ is $p$-nilpotent.
Suppose that $p$ is a prime. Let $G$ be a group and $P$ a Sylow $p$-subgroup of $G$. Assume that $N_G(P)$ is $p$-nilpotent. Then $G$ is $p$-nilpotent if one of the following holds:

1. $\Omega(P) \leq Z_{p-1}(P)$.
2. when $p = 2$, $\Omega_1(P) \leq Z(P)$ and $P$ is quaternion-free.
Remark J González-Sánchez and T. S. Weigel apply cohomology ring theory to obtain the result. Ballester-Bolinches, etc. give another approach based on the classical theory of Hall and Higman (see B. Huppert and N. Blackburn, Finite Groups II, Chapter IX).
We mention the following result:

**A. Ballester-Bolinches and X. Guo, J. Algebra, 2000**

Let $p$ be a prime dividing the order of a group $G$ and let $P$ be a Sylow $p$-subgroup of $G$. Assume that $N_G(P)$ is $p$-nilpotent. Then $G$ is $p$-nilpotent if one of the following holds:

1. $\Omega(P \cap G') \leq Z(P)$;
2. when $p = 2$, $\Omega_1(P \cap G') \leq Z(P)$ and $P$ is quaternion-free.

where $G'$ is the commutator subgroup of $G$. 

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Generalization I of Burnside Theorem

Remark

- $P \cap O^p(G) \leq P \cap G'$ if $P$ is a Sylow $p$-subgroup of $G$;
- $G$ is $p$-nilpotent if and only if $P \cap O^p(G) = 1$. 
Li, Su, Wang, Xie’s Theorem 1

Suppose that \( p \) is a prime. Let \( G \) be a group and \( P \) a Sylow \( p \)-subgroup of \( G \). Assume that \( N_G(P) \) is \( p \)-nilpotent. Then \( G \) is \( p \)-nilpotent if and only if one of the following holds:

1. \( \Omega(P \cap O^p(G)) \leq Z_{p-1}(P) \).
2. when \( p = 2 \), \( \Omega_1(P \cap O^p(G)) \leq Z(P) \) and \( P \) is quaternion-free.
Generalization I of Burnside Theorem

The works in the above is to weaken the condition “ $P \leq Z(P)$”:

1. enlarge $Z(P)$: $Z(P) \rightarrow Z_{p-1}(P)$;
2. lessen $P$: $P \rightarrow \Omega(P) \rightarrow \Omega(P \cap G') \rightarrow \Omega(P \cap O^p(G)) \rightarrow \Omega(P \cap G^\wedge) \rightarrow \Omega(P \cap G^L)$.

Where $G^\wedge$ is the nilpotent residual of $G$, $G^L$ is the $L$-residual of $G$, $L$ is the class of all $p$-solvable groups whose $p$-lengths are at most 1.
From the other view to extend Burnside’s Theorem. We first mention two results in this line, one belongs to Wielandt, the other belongs to Ballester-Bolinches and Esteban-Romero.
A $p$-group $G$ is called *regular* if, for any $x, y \in G$, there holds the following:

$$x^p y^p = (xy)^p \prod d_i^p,$$

where $d_i \in \langle x, y \rangle'$. 

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Some generalizations of Burnside’s Theorem
A $p$-group is regular if its nilpotency class is less than $p$. Hence we can see that Wielandt’s following result is a generalization of Hall’s, of course, Burnside’s Theorem.


Let $p$ be a fixed prime in $\pi(G)$ and $P$ a Sylow $p$-subgroup of $G$. Suppose that $N_G(P)$ is $p$-nilpotent. Then $G$ is $p$-nilpotent if $P$ is regular.
A group $G$ is called *modular* if, for any subgroups $H$ and $K$ of $G$, $HK$ is a subgroup of $G$. 
A. Ballester-Bolinches and R. Esteban-Romero, J. Algebra, 2002

Let $p$ be a fixed prime in $\pi(G)$ and $P$ a Sylow $p$-subgroup of $G$. Suppose that $N_G(P)$ is $p$-nilpotent. Then $G$ is $p$-nilpotent if $P$ is modular.
Generalization II of Burnside Theorem

Hall’s result and the above result also follow from Yoshida’s transfer theorem:

Yoshida’s theorem  
The normalizer of a Sylow $p$-subgroup $P$ of a group $G$ controls $p$-transfer unless $P$ has a homomorphic image isomorphic to $C_p \rtimes C_p$, where $\rtimes$ is the wreath product.
Questions

1. Does the theorem hold if $P$ is other kind of $p$-group? For example, meta-cyclic, meta-abelian, $p$-group of maximal class, Powerful $p$-group, etc. the answer is no! see Example 2 in the last section.

2. Suppose that $P$ is meta-cyclic. If $p > 2$, a meta-cyclic $p$-group is $p$-regular, hence the answer is yes; but in the case $p = 2$, the answer is no! $S_4$ is a counter-example.

3. Does the theorem hold if $P \cap O^p(G)$ is a special kind of $p$-group? The answer is no! see Example 1 in the last section.
I think there exists a generalized regular property of $p$-group or a generalized modular property of $p$-group such that, under the assumption that $N_G(P)$ is $p$-nilpotent, $G$ is $p$-nilpotent if and only if $P$ possesses this property.
"P is abelian" ⇔ "P′ = 1"

From this point of view, we have:

Li, Su, Wang, Xie’s Theorem 2

Let p be a fixed prime in \( \pi(G) \) and \( P \) a Sylow p-subgroup of \( G \). Suppose that \( N_G(P) \) is \( p \)-nilpotent. Then \( G \) is \( p \)-nilpotent if \( P' \) is normal in \( G \).
Generalization III of Burnside Theorem

Go further:

**Li, Su, Wang, Xie’s Theorem 3**

Let \( p \) be a fixed prime in \( \pi(G) \) and \( P \) a Sylow \( p \)-subgroup of \( G \). Suppose that \( N_G(P) \) is \( p \)-nilpotent. Then \( G \) is \( p \)-nilpotent if \( P' \) is \( s \)-permutable in \( G \).

**Corollary ( X. Guo and X. Zhao, Acta Mathematica Scientia, 2008)**

Let \( p \) be the smallest prime dividing the order of a group \( G \) and \( P \) a Sylow \( p \)-subgroup of \( G \). If every maximal subgroup of \( P \) is \( s \)-permutable in \( N_G(P) \) and \( P' \) is \( s \)-permutable in \( G \), then \( G \) is \( p \)-nilpotent.
Known results:

- Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ a Sylow $p$-subgroup of $G$. If every maximal subgroup of $P$ is s-permutable in $G$, then $G$ is $p$-nilpotent.

- (Schmid, J Algebra, 1998) if $H$ is a normal subgroup of a Sylow subgroup of a group $G$ and $H$ is s-permutable in $G$, then $H$ is normal in $G$. 
Similarly, we consider other generalized normalities.

**Li, Su, Wang, Xie’s Theorem 4**

Let $p$ be a fixed prime in $\pi(G)$ and $P$ a Sylow $p$-subgroup of $G$. Suppose that $N_G(P)$ is $p$-nilpotent. Then $G$ is $p$-nilpotent if and only if $P'$ is a partial CAP-subgroup of $G$. 
Known result:
$G$ is $p$-nilpotent if every maximal subgroup of $P$ is a CAP-subgroup of $G$, where $p$ is the smallest prime in $\pi(G)$ and $P$ is a Sylow $p$-subgroup of $G$.

Let $p$ be the smallest prime in $\pi(G)$ and $P$ a Sylow $p$-subgroup of $G$. Then $G$ is $p$-nilpotent if every maximal subgroup of $P$ is a CAP-subgroup of $N_G(P)$ and $P'$ is a CAP-subgroup of $G$. 
Naturally, we have the following questions:

1. Can we replace the subgroup $P'$ by other kinds of subgroups of $P$ in the above theorems? For example, $\Phi(P)$, etc., etc.

2. Can we unify the above theorems.
We have the following result which unify the above theorem and Hall’s result:

**Li, Su, Wang and Xie’s Theorem 5**

Let \( p \) be a fixed prime in \( \pi(G) \) and \( P \) a Sylow \( p \)-subgroup of \( G \). Suppose that \( N_G(P) \) is \( p \)-nilpotent. Then \( G \) is \( p \)-nilpotent if and only if there exists a normal subgroup \( N \) of \( P \) contained in \( \Phi(P) \) such that \( N \) is a partial CAP-subgroup of \( G \) and the nilpotency class of \( P/N \) is less than \( p \), i.e., \( P/N \leq Z_{p-1}(P/N) \).
Remark

1. Any normal subgroup $N$ of $P$ with $P' \leq N \leq \Phi(P)$ satisfies the conditions in the above theorem: the nilpotency class of $P/N$ is less than $p$. Hence this condition is not much demanding.

2. If $G$ is $p$-nilpotent, then every $p$-subgroup of $G$ is a CAP subgroup of $G$. So the condition that $N$ is a partial CAP-subgroup of $G$ is natural.

3. If $P \leq Z_{p-1}(P)$, then pick $N = 1$. We obtain Hall's theorem; If $P'$ is a CAP-subgroup of $G$, $P/P'$ is abelian. Hence we obtain our Theorem 4.
We also have some conjectures:

**Conjecture 1** Let $p$ be a fixed prime in $\pi(G)$ and $P$ a Sylow $p$-subgroup of $G$. Suppose that $N_G(P)$ is $p$-nilpotent. Then $G$ is $p$-nilpotent if and only if there exists a normal subgroup $N$ of $P$ contained in $\Phi(P)$ such that $N$ is a partial CAP-subgroup of $G$ and $P/N$ satisfies the following:

1. $\Omega(P/N) \leq Z_{p-1}(P/N)$.
2. When $p = 2$, $\Omega_1(P/N) \leq Z(P/N)$ and $P/N$ is quaternion-free.
**Conjecture 2**  Let $p$ be a fixed prime in $\pi(G)$ and $P$ a Sylow $p$-subgroup of $G$. Suppose that $N_G(P)$ is $p$-nilpotent. Then $G$ is $p$-nilpotent if and only if there exists a normal subgroup $N$ of $P$ such that $N \trianglelefteq \Phi(P)$, $P/N$ is a modular $p$-group and $N$ is a partial CAP-subgroup of $G$. 
We know that $G$ is $p$-nilpotent iff $P \cap O^p(G) = 1$.
The following example shows that $G$ may not be $p$-nilpotent if we assume that $P \cap O^p(G)$ is abelian or normal in $G$, under the hypothesis that $N_G(P)$ is $p$-nilpotent.
Example 1
Let $H = A_4 = C_3[B_4]$ (the alternating group on 4 symbols). This has a faithful irreducible module of dimension 3 over the field of 3 elements $GF(3)$. Call this module $N = C_3 \times C_3 \times C_3$. Let $G = [N]H = [N](C_3[B_4])$.
Assume that $p = 3$.
Then $G_3 = P = [N]C_3$, $O^3(G) = NB_4$.
$P \cap O^3(G) = N$ is abelian and normal in $G$.
But $G$ is not 3-nilpotent.
Example 2 Suppose that $G = S_4$, the symmetric group of degree 4, $p = 2$, $P$ is a Sylow 2-group, then $N_G(P) = P = D_8$, $D_8$ is meta-cyclic, but $G$ is not 2-nilpotent.
Final Remark

Thanks for all presenting here!