Smooth renormings of the Lebesgue-Bochner space $L^1(\mu, X)$

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Theorem 1

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$$\varphi(f) = \int_\Omega M\left(\|f(\omega)\|\right) \mathrm{d}\mu(\omega), \quad f \in L^1(\mu, X). \tag{1}$$

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If the norm $\|\cdot\|$ on X is (uniformly) Gâteaux smooth, then the function φ defined by (1) is (uniformly) Gâteaux differentiable on $L^1(\mu, X)$, and for all $f, h \in L^1(\mu, X)$ we have

$$\varphi'(f)h=\int_{\Omega} M'\left(\|f(\omega)\|\right)\|\cdot\|'\left(f(\omega)\right)\left(h(\omega)\right) \mathrm{d}\mu(\omega). \tag{2}$$

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Convention $\|\cdot\|'(0) = 0$.

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Corollary 4

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Corollary 4

(Rosenthal) Let (Ω, Σ, μ) be a probability space. If X is a super-reflexive Banach space, then every reflexive subspace of $L^1(\mu, X)$ is super-reflexive.

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$$\limsup_{t \to 0} \left\{ \left| \frac{\varphi(f + th) - \varphi(f)}{t} - \varphi'(f)h \right| : \; h \in B_Y \right\} = 0.$$

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In particular, φ *restricted to* Y, *is then Fréchet differentiable on* Y.

Proposition 6

Let $(X, \|\cdot\|)$ *be a Banach space and* (Ω, Σ, μ) *be a probability space. Let* Y *be a reflexive subspace of* $L^1(\mu, X)$*. If the norm* $\|\cdot\|$ *is uniformly Fréchet smooth, then*

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Let $(X, \|\cdot\|)$ *be a Banach space and* (Ω, Σ, μ) *be a probability space. Let* Y *be a reflexive subspace of* $L^1(\mu, X)$*. If the norm* $\|\cdot\|$ *is uniformly Fréchet smooth, then the derivative* φ' *is "*Y*-uniformly continuous", that is,*

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$$|\varphi'(f)h - \varphi'(g)h| < \varepsilon$$

whenever $f, g \in L^1(\mu, X),\ h \in B_Y$*, and* $\|f - g\|_{L^1(\mu,X)} < \delta$*.*

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Let $(X, \|\cdot\|)$ be a Banach space and (Ω, Σ, μ) be a probability space. Let Y be a reflexive subspace of $L^1(\mu, X)$. If the norm $\|\cdot\|$ is uniformly Fréchet smooth, then the derivative φ' is "Y-uniformly continuous", that is, for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|\varphi'(f)h - \varphi'(g)h| < \varepsilon$$

whenever $f, g \in L^1(\mu, X)$, $h \in B_Y$, and $\|f - g\|_{L^1(\mu,X)} < \delta$. In particular, φ restricted to Y is then uniformly Fréchet differentiable on Y.

THANK YOU

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