Shadowing of pseudo-orbits and Furstenberg families

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Basic notation

1. (X, d) - compact metric space
2. $f : X \to X$ - continuous
3. f is continuous (doesn't have to be invertible or onto)
4. (X, f) - transitive - if for every open $U, V \neq \emptyset$ there is $n > 0$ such that $f^n(U) \cap V \neq \emptyset$.
5. (X, f) - (topologically) mixing - if for every open $U, V \neq \emptyset$ there is $N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$.

Shadowing property

1. A sequence $\{x_i\}_{i=0}^{\infty}$ is a δ-pseudo-orbit if $d(f(x_i), x_{i+1}) < \delta$ for all $i \geq 0$.
2. A δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ is ε-traced by a point $z \in X$ when $d(f^n(z), x_n) < \varepsilon$ for $n = 0, 1, \ldots$.
3. f has shadowing property if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit is ε-traced.

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Remark

Tracing of finite pseudo orbits is an equivalent property.

Chain-recurrent set

1. x is chain-recurrent if for every $\delta > 0$ there is δ-pseudo orbit $x_0, x_1, \ldots, x_n$ from x to x (i.e. $x_0 = x_n = x$, $d(f(x_i), x_{i+1}) < \delta$).
2. $CR(f)$ - the set of chain-recurrent points.

Theorem

Chain-recurrent set is internally chain-recurrent, that is

1. $CR(f) = CR(f|_{CR(f)})$,

If f has the shadowing property then:

1. $f(CR(f)) = CR(f)$ *(and $CR(f)$ is always closed),*
2. $f|_{CR(f)}$ *has shadowing,*
3. $CR(f) = \Omega(f) = \overline{Rec(f)}$.

Shadowing (general idea as it is defined)

1. For δ we define δ-pseudo-orbit (in some sense)
2. For ε we define ε-tracing (in some sense) of pseudo-orbit by a point
3. Then shadowing property means that for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit is ε-traced (where tracing nad pseudo-orbit are defined accordingly).
4. For limit shadowing, we need to define asymptotic pseudo-orbit and asymptotic tracing.

Furstenberg families

1. $\mathcal{F} \subset P(\mathbb{N})$ is Fursenberg family (hereditarily upward set, i.e. $A \in \mathcal{F}$, $A \subset B$ then also $B \in \mathcal{F}$).
2. For $J \subset \mathbb{N}$ we define its lower and upper (asymptotic) density:
   - $D_*(A) = \liminf_{n\to\infty} \frac{1}{n}\#(A \cap [0, n))$
   - $D^*(A) = \limsup_{n\to\infty} \frac{1}{n}\#(A \cap [0, n))$
3. Some standard families:
   - $\mathcal{D} = \{A \subset \mathbb{N} : D_*(A) = 1\}$
   - $\mathcal{F}_t$ consists of thick sets

$$(\text{i.e. } T \in \mathcal{F}_t \iff \forall\, n\, \exists\, i\ \{i, i+1, \dots, i+n\} \subset T).$$

Partial shadowing

1. A dynamical system (X, f) has $(\mathcal{F}, \mathcal{G})$-shadowing if
   - for every $\varepsilon > 0$ there is $\delta > 0$ such that
   - for every sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that

$$\{n : d(f(x_n), x_{n+1}) < \delta\} \in \mathcal{F}$$

   - there is $z \in X$ such that

$$\{n : d(f^n(z), x_n) < \varepsilon\} \in \mathcal{G}$$

2. Examples:
   - $(\{\mathbb{N}\}, \{\mathbb{N}\})$ -shadowing $\equiv$ "standard" shadowing
   - $(\mathcal{D}, \mathcal{D})$ -shadowing $\equiv$ so-called ergodic shadowing

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Theorem (Fakhari and Ghane, 2010)

The following properties are equivalent for (X, f):

1. $(\mathcal{D}, \mathcal{D})$ *-shadowing property,*
2. *topological mixing with shadowing property.*

Thick shadowing

- $(\{\mathbb{N}\}, \{\mathbb{N}\})$ -shadowing $\equiv$ "standard" shadowing
- $(\mathcal{D}, \mathcal{D})$ -shadowing $\equiv$ ergodic shadowing
- $(\mathcal{D}, \mathcal{F}_t)$ -shadowing $\equiv$ thick shadowing [Dastjerdi & Hosseini,2010]
  - recall that $\mathcal{F}_t$ consists of thick sets
  - (i.e. $T \in \mathcal{F}_t \iff \forall\, n\, \exists\, i\; \{i, i+1, \dots, i+n\} \subset T$).

Thick shadowing and chain recurrence (Brian, Meddaugh and Raines, 2014)

Theorem (B., M. & R.)

If (X, f) is transitive then the following properties are equivalent:

1. *the shadowing property,*
2. *the thick shadowing property (i.e. $(\mathcal{D}, \mathcal{F}_t)$-shadowing),*
3. *the $(\mathcal{F}_t, \mathcal{F}_t)$-shadowing property,*
4. *the $(\{\mathbb{N}\}, \mathcal{F}_t)$-shadowing property.*

Example

Map $f(x) = x + \frac{1}{4}|\sin(\pi x)|$ on $[0, 2]$ has thick shadowing but not shadowing.

Question (B., M. & R.)

Does shadowing imply thick shadowing?

Furstenberg families (a few more)

1. $\mathcal{F}_t$ consists of thick sets

   (i.e. $T \in \mathcal{F}_t \iff \forall\, n\, \exists\, i\ \{i, i+1, \ldots, i+n\} \subset T$).
2. $\mathcal{F}_s = \{A : \forall T \in \mathcal{F}_t, \quad A \cap T \neq \emptyset\}$ consists of syndetic sets.
3. $\mathcal{F}_{cf} = \{A : \#(X \setminus A) < \infty\}$ consists of co-finite sets.
4. $\mathcal{F}_{ps} = \{S \cap T : S \in \mathcal{F}_s, T \in \mathcal{F}_t\}$ – piecewise syndetic sets.
5. A family $\mathcal{F}$ has partition regularity or the Ramsey property if, whenever $F \in \mathcal{F}$ and $F = \cup_{i=1}^{n} F_i$ then $F_j \in \mathcal{F}$ for some j.

Shadowing and thick shadowing

Theorem (O.)

For every dynamical system (X, f) the following conditions are equivalent:

1. $CR(f) = \Omega(f) = \overline{Rec(f)}$ *and the dynamical system* $(CR(f), f|_{CR(f)})$ *has the shadowing property,*
2. (X, f) *has the thick shadowing property,*
3. (X, f) *has the* $(\mathcal{F}_t, \mathcal{F}_t)$*-shadowing property,*
4. (X, f) *has the* $(\{\mathbb{N}\}, \mathcal{F}_t)$*-shadowing property,*
5. (X, f) *has the* $(\{\mathbb{N}\}, \mathcal{F}_{cf})$*-shadowing property.*

Shadowing and thick shadowing - further properties

Theorem (O.)

If (X, f) is chain-recurrent (i.e. $CR(f) = X$) then the following statements are equivalent:

1. *(X, f) has the thick shadowing property,*
2. *(X, f) has the shadowing property.*

Corollary

The following statements are equivalent:

1. (X, f) has the thick shadowing property,
2. $(CR(f), f|_{CR(f)})$ has the thick shadowing property,

Families with Ramsey property

Theorem (O.)

The following conditions are equivalent:

1. *minimal points are dense in* X,
2. (X, f) *has the* $(P(\mathbb{N}), \mathcal{F})$*-shadowing property for every family* $\mathcal{F}$ *with the Ramsey property,*
3. (X, f) *has the* $(P(\mathbb{N}), \mathcal{F}_{ps})$*-shadowing property.*

Average shadowing property (1988, Blank)

1. $\{x_i\}_{i=0}^{\infty}$ is a δ-average pseudo-orbit of f if there is a number N such that for all $n > N$ and $k \geq 0$
$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$
2. f has the average shadowing property if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-average pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is ε-shadowed on average by some point $z \in X$, that is:
$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \varepsilon.$$

Some relations (work with D. A. Dastjerdi and M. Hosseini)

1. Average shadowing property (ASP) implies $(\mathcal{D}, \mathcal{F}_{\underline{d}})$-shadowing,where
$$\mathcal{F}_{\underline{d}} = \left\{ A \subset \mathbb{N} : \liminf_{n\to\infty} \frac{1}{n} \#(A \cap [0,n)) > 0 \right\}.$$
2. If (X, f) has shadowing, then (X, f) has average shadowing iff (X, f) is mixing.

Some relations (work with D. A. Dastjerdi and M. Hosseini)

1. Average shadowing property (ASP) implies $(\mathcal{D}, \mathcal{F}_{\underline{d}})$-shadowing,
2. If (X, f) has shadowing, then (X, f) has average shadowing iff (X, f) is mixing.
3. If (X, f) is minimal and has shadowing then it is an adding machine (odometer).
4. If (X, f) is minimal and has $(\mathcal{D}, \mathcal{F}_{\underline{d}})$-shadowing then it is weakly mixing. Every minimal homeomorphism with u.p.e (of all orders) has $(\mathcal{D}, \mathcal{F}_{\underline{d}})$-shadowing.
5. minimal weakly mixing (X, f) never has $(\mathcal{D}, \mathcal{F}_{s})$-shadowing

Question

Is there a full characterization of minimal systems with $(\mathcal{D}, \mathcal{F}_{\underline{d}})$-shadowing.

Related questions

1. (X, f) is c.p.e. if every factor has positive topological entropy,
2. there is transitive (X, f) with c.p.e. but without $(\mathcal{D}, \mathcal{F}_{\underline{d}})$-shadowing

Question

Is there minimal systems with c.p.e. but without $(\mathcal{D}, \mathcal{F}_{\underline{d}})$-shadowing.

Question

Is there a nontrivial minimal system with the ASP?

Some special Furstenberg families

1. Recall:
   - $D_*(A) = \liminf_{n\to\infty} \frac{1}{n}\#(A \cap [0, n))$
   - $D^*(A) = \limsup_{n\to\infty} \frac{1}{n}\#(A \cap [0, n))$
2. For each $\alpha > 0$ define:
   - $\mathcal{M}^\alpha = \{A \subset \mathbb{N} : D^*(A) > \alpha\}$
   - $\mathcal{M}_\alpha = \{A \subset \mathbb{N} : D_*(A) > \alpha\}$

Characterization of ASP (work with X. Wu and G. Chen)

Theorem

Let (X, f) be a dynamical system. Then the following statements are equivalent:

1. *f has the average shadowing property,*
2. *f has the $(\mathcal{D}, \mathcal{M}_\alpha)$-shadowing property for every $\alpha \in [0, 1)$.*

Remark

With this result at hand it becomes clear why ASP implies weak mixing on measure center $M = \bigcup_\mu \text{supp}\mu$.

Remark

Minimal system with ASP is weakly mixing.

Almost specification (very rough approximation of definition)

Map f has almost specification if for any sequence of integers $n_1, \ldots, n_k$ and numbers $\varepsilon_1, \ldots, \varepsilon_k > 0$:

1. for every sequence of orbits

$$x_1, f(x_1), \ldots, f^{n_1}(x_1), x_2, f(x_2), \ldots, x_k, f(x_k), \ldots, f^{n_k}(x_k)$$

2. there is a tracing point z which is further than ε from appropriate segment of orbit at no more than $g(n_i, \varepsilon_i)$ positions, where $g(\cdot, \cdot)$ is a function (depending only on f) such that $\lim_n g(n, \varepsilon)/n = 0$.

Relations between ASP and almost specification

- In 2012 we found a partial answer

Theorem (Kwietniak, Kulczycki & O.)

almost specification $\Longrightarrow$ *ASP (when f is onto)*

Relations between ASP and almost specification

- In 2012 (2014) we found a partial answer

Theorem (Kwietniak, Kulczycki & O.; Chen, O., Wu)

almost specification $\Longrightarrow$ *ASP*

Relations between ASP and almost specification

- In 2012 we found a partial answer

Theorem (Kwietniak, Kulczycki & O.; Wu, O., Chen)

almost specification $\Longrightarrow$ *ASP*

- Recently we were able to make a progress (joint work with D. Kwietniak and M. Łącka, in progress...)

Theorem (Kwietniak, Łącka, O.)

ASP $\not\Longrightarrow$ *almost specification.*

Why almost specification? - β-shifts

- β-transformation ($\beta > 1$):

$$T_\beta \colon x \mapsto \beta x \pmod 1$$

- β-shift X_β - shift defined by natural partition for T_β.

Theorem (Pfisfer & Sullivan, 2005; Buzzi, 1997)

Every (X_β, σ) has the almost specification property, but for some β (in fact set of full Lebesgue measure) the specification property is not satisfied.

Theorem (Pfisfer & Sullivan, 2007)

Specification $\Longrightarrow$ Almost specification

Measure center

Theorem (Kulczycki, Kwietniak & O.)

If A contains the measure center and $f|_A$ has almost specification (resp. ASP) then f also has almost specification (resp. ASP)

1. There are proximal systems (X, f) (and with singleton measure center) such that
   1. (X, f) is transitive but not weakly mixing; or
   2. (X, f) is weakly mixing but not mixing; or
   3. (X, f) is mixing (but obviously cannot have specification property);

Measure center

Theorem (Kulczycki, Kwietniak & O.)

If A contains the measure center and $f|_A$ has almost specification (resp. ASP) then f also has almost specification (resp. ASP)

Theorem (Wu, O. & Chen)

If f has almost specification then $f|_A$ has almost specification

Question

Is the same true for ASP?

1. There are proximal systems (X, f) (and with singleton measure center) such that
   1. (X, f) is transitive but not weakly mixing; or
   2. (X, f) is weakly mixing but not mixing; or
   3. (X, f) is mixing (but obviously cannot have specification property);

Two results with Kwietniak and Rams

Theorem

If (X, T) has the almost specification property restricted to the measure center then

- *it is topological K.*
- *if the measure center is non-trivial, then (X,T) has a horseshoe.*

Corollary

Minimal system with almost specification is trivial.

Theorem

There is a subshift with almost specification and more than one measure of maximal entropy.

THE END

Thank you for your attention!