Sequence spaces of type $S^\nu$, and beyond with wavelet leaders

Céline Esser

Context

► Study of the pointwise regularity of a signal $f$ by means of the Hölder exponents $h_f$

$$h_f(x_0) = \sup \{\alpha \geq 0 : f \in C^\alpha(x_0)\}$$

and multifractal formalisms, which are formulas that are expected to yield the spectrum of singularities of $f$ defined by

$$d_f(h) = \dim_H \{x \in \mathbb{R} : h_f(x) = h\}.$$  

► Signals of $L^2([0, 1])$ are represented through their wavelet coefficients

$$f = \sum_{j \geq 0} \sum_{k=0}^{2^j-1} c_{j,k} \psi_{j,k}$$

and the Hölder regularity can be characterized using this representation.
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The classical use of Besov spaces leads to a loss of information (for example, only the concave hull and the increasing parts of the spectrum can be recovered).

S. Jaffard introduced spaces of type $S^\nu$

Aim : Detection of non concave spectrum.

More recently, introduction of spaces of the same type but based on the wavelet leaders of the signal

Aim : Detection of non increasing spectrum and of the oscillating singularities.
Let $c$ be the sequence of wavelet coefficients of $f$. The wavelet profile $\nu_f$ of $f$ is defined by

$$\nu_f(\alpha) = \lim_{\varepsilon \to 0^+} \left( \limsup_{j \to +\infty} \left( \frac{\log \#E_j(1, \alpha + \varepsilon)(f)}{\log 2^j} \right) \right)$$

for all $\alpha \in \mathbb{R}$, where

$$E_j(C, \alpha)(f) = \{ k : |c_{j,k}| \geq C 2^{-\alpha j} \} .$$

**Interpretation:** there are "approximatively" $2^{\nu_f(\alpha)j}$ coefficients greater in modulus than $2^{-\alpha j}$.

** ν_f is independant of the chosen wavelet basis.**
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- **Interpretation**: there are "approximatively" \( 2^{\nu_f(\alpha)j} \) coefficients greater in modulus than \( 2^{-\alpha j} \).
- \( \nu_f \) is independant of the chosen wavelet basis.
Consider an increasing function $\nu : \mathbb{R} \to \{-\infty\} \cup [0, 1]$ right continuous (called an admissible profile). Define

$$\alpha_{\text{min}} := \inf \{\alpha : \nu(\alpha) \geq 0\}$$

$$\alpha_{\text{max}} := \inf \{\alpha : \nu(\alpha) = 1\}.$$ 

Denote $\Omega$ the set of complex sequences $c = (c_{j,k})_{j \in \mathbb{N}, k \in \{0, \ldots, 2^j - 1\}}$.

**Définition**

The space $S^\nu$ is the set of sequences $c \in \Omega$ such that

$$\forall \alpha \in \mathbb{R}, \forall \varepsilon > 0, \forall C > 0, \exists J \geq 0 : \# E_j(C, \alpha)(c) \leq 2^{(\nu(\alpha) + \varepsilon)j}, \forall j \geq J$$

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The space \( S^{\nu} \) is the set of sequences \( c \in \Omega \) such that

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where

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E_j(C, \alpha)(c) = \{ k : |c_{j,k}| \geq C2^{-\alpha j} \}.
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Proposition

The space $S^\nu$ is a vector space and

$$S^\nu = \{ c \in \Omega : \nu_c(\alpha) \leq \nu(\alpha), \forall \alpha \in \mathbb{R} \}.$$ 

Examples of $S^\nu$ spaces:

- Assume that $\nu(\alpha) = 1$ for all $\alpha \in \mathbb{R}$. If $c \in \Omega$, for every $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C' > 0$,

$$\#E_j(C, \alpha)(c) \leq 2^j < 2^{(\nu(\alpha)+\varepsilon)j} \quad \forall j \in \mathbb{N}.$$ 

This means that $c \in S^\nu$ and therefore, $S^\nu = \Omega$.

- Assume that

$$\nu(\alpha) = \begin{cases} -\infty & \text{si } \alpha < a \\ 1 & \text{si } \alpha \geq a \end{cases}$$

where $a \in \mathbb{R}$. Then $S^\nu$ is the set of sequences $c$ such that for every $\alpha < a$,

$$\sup_{j \in \mathbb{N}} \sup_{k \in \{0, \ldots, 2^j - 1\}} 2^{\alpha j} |c_{j,k}| < +\infty.$$
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Besov Spaces

For $s \in \mathbb{R}$ and $p > 0$, a function belongs to the Besov space $b^s_{p,\infty}$ if its wavelets coefficients satisfy

$$
\|c\|_{b^s_{p,\infty}} := \sup_{j \in \mathbb{N}_0} 2^{(s - \frac{1}{p})j} \left( \sum_{k=0}^{2^j - 1} |c_{j,k}|^p \right)^{\frac{1}{p}} < +\infty.
$$

The definition is extended to the case $p = \infty$ by setting

$$
\|c\|_{C^s} := \sup_{j \in \mathbb{N}_0} \sup_{k \in \{0, \ldots, 2^j - 1\}} 2^{sj} |c_{j,k}|.
$$

This corresponds to the Hölder space of order $s$, denoted by $C^s$. These spaces are independent of the wavelet mother chosen. Considered as sequence spaces, they are Banach spaces if $p \geq 1$ and complete metric spaces if $p < 1$. 
If we define the concave conjugate $\eta$ of $\nu$ by

$$\eta(p) := \inf_{\alpha \geq \alpha_{\min}} (\alpha p - \nu(\alpha) + 1)$$

we get the following characterization of $S^\nu$ spaces.

**Link with Besov Spaces**

If $(p_n)_{n \in \mathbb{N}}$ is a dense sequence of $]0, +\infty[$ and if $(\varepsilon_m)_{m \in \mathbb{N}}$ is a sequence of strictly positive numbers converging to 0, then

$$S^\nu \subset \bigcap_{\varepsilon > 0} \bigcap_{p > 0} b_{p, \infty}^{\eta(p)/p - \varepsilon} = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} b_{p_n, \infty}^{\eta(p_n)/p_n - \varepsilon_m}$$

and this inclusion becomes an equality if and only if $\nu$ is concave.
Definition
Let $\alpha \in \mathbb{R}$ and $\beta \in \{-\infty\} \cup [0, +\infty[$. A sequence $c$ belongs to the auxiliary space $E(\alpha, \beta)$ if there exists $C, C' \geq 0$ such that

$$\# \left\{ k : |c_{j,k}| \geq C2^{-\alpha j} \right\} \leq C'2^{\beta j}, \quad \forall j \geq 0.$$ 

Proposition
For any sequence $(\alpha_n)_{n \in \mathbb{N}}$ dense in $\mathbb{R}$ and $\forall (\varepsilon_m)_{m \in \mathbb{N}} \to 0^+$,

$$S^\nu = \bigcap_{\varepsilon > 0} \bigcap_{\alpha \in \mathbb{R}} E(\alpha, \nu(\alpha) + \varepsilon) = \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} E(\alpha_n, \nu(\alpha_n) + \varepsilon_m).$$
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The definition on a distance on every auxiliary spaces $E(\alpha, \nu(\alpha) + \varepsilon)$ will provide a distance on $S'_\nu$ (initial topology).

**Classical result of functional analysis**

Let $(E_m)_{m \in \mathbb{N}}$ be a sequence of spaces endowed with the topologies defined by the distances $d_m$, and let $E = \bigcap_{m \in \mathbb{N}} E_m$. On $E$, let us consider the topology defined as follows: for every $e \in E$, a basis of neighbourhoods of $e$ is given by the family of sets

$$\bigcap_{(m)} \{ f \in E : d_m(e, f) \leq r_m \}, \quad r_m > 0.$$  

Then,

1. This topology is equivalent to the topology defined on $E$ by the distance $d$ given by

$$d(e, f) = \sum_{m=1}^{+\infty} 2^{-m} \frac{d_m(e, f)}{1 + d_m(e, f)}, \quad e, f \in E.$$  

2. A sequence is a Cauchy sequence in $(E, d)$ if and only if it is a Cauchy sequence in $(E_m, d_m)$ for every $m \in \mathbb{N}$.

3. A sequence converges to $e$ in $(E, d)$ if and only if it converges to $e$ in $(E_m, d_m)$ for every $m \in \mathbb{N}$.  

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Distance on the auxiliary space $E(\alpha, \beta)$

$$\delta_{\alpha, \beta}(c, c') := \inf \left\{ C + C' : C, C' \geq 0 \text{ and } \#E_j(C, \alpha)(c - c') \leq C'2^{\beta j} \forall j \in \mathbb{N} \right\}$$

From the characterization of $S^{\nu}$ as an intersection of auxiliary spaces, we get a distance $d$ on $S^{\nu}$.

Property

The space $(S^{\nu}, d)$ is a complete separable topological vector space.
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From the characterization of $S^\nu$ as an intersection of auxiliary spaces, we get a distance $d$ on $S^\nu$.

**Property**

The space $(S^\nu, d)$ is a complete separable topological vector space.
Definition
Let $0 < p \leq 1$. A subset $K$ of a vector space $E$ is $p$-convex if for every $x_1, \ldots, x_N \in K$ and for every $\theta_1, \ldots, \theta_N \geq 0$ such that $\sum_{n=1}^{N} \theta_n^p = 1$, then $p$-convex combinaison $\sum_{n=1}^{N} \theta_n x_n$ belongs to $K$. The set $K$ is absolutely $p$-convex if it is $p$-convex and if

\[ \forall x \in K, \forall |\lambda| \leq 1, \lambda x \in K. \]

Proposition
Let $0 < p \leq 1$. A subset $K$ of a vector space $E$ is absolutely $p$-convex if and only if

\[ \sum_{i=1}^{n} \mu_i K \subset K \]

for all $\mu_1, \ldots, \mu_n \in \mathbb{C}$ such that $\sum_{i=1}^{n} |\mu_i|^p \leq 1$. 
**p local convexity**

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Definition
A topological vector space is $p$-locally convex if it has a basis of 0-nbh absolutely $p$-convex.

Definition
Let $E$ be a vector space and let $0 < p \leq 1$. An application $q : E \to [0, +\infty[$ is a $p$ semi-norm if

\[
\begin{align*}
q(\lambda e) &= |\lambda| q(e) \\
(q(e + f))^p &\leq (q(e))^p + (q(f))^p
\end{align*}
\]

$\forall \lambda \in \mathbb{C}, e \in E$

$\forall e, f \in E$

If moreover, $q(e) = 0 \iff e = 0$, then $q$ is a $p$ norm.

Theorem
A topological vector space $(E, \mathcal{T})$ is $p$-locally convex if and only if there exists a family of $p$ semi-norms on $E$ that defined a topology equivalent to the topology $\mathcal{T}$.
**Definition**

A topological vector space is *p*-locally convex if it has a basis of 0-nbh absolutely *p*-convex.

**Definition**

Let $E$ be a vector space and let $0 < p \leq 1$. An application $q : E \to [0, +\infty[$ is a *p* semi-norm if

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**Theorem**

A topological vector space $(E, T)$ is *p*-locally convex if and only if there exists a family of *p* semi-norms on $E$ that defined a topology equivalent to the topology $T$. 
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If moreover, $q(e) = 0 \iff e = 0$, then $q$ is a *p* norm.

**Theorem**
A topological vector space $(E, \mathcal{T})$ is *p*-locally convex if and only if there exists a family of *p* semi-norms on $E$ that defined a topology equivalent to the topology $\mathcal{T}$. 
Definition

We define
\[ \partial^+ \nu(\alpha) := \liminf_{h \to 0^+} \frac{\nu(\alpha + h) - \nu(\alpha)}{h} . \]
for every \( \alpha \in \mathbb{R} \) such that \( \alpha \geq \alpha_{\min} \). The local convexity index of \( \nu \) is defined by
\[ p_0 := \min \left( 1, \inf_{\alpha_{\min} \leq \alpha < \alpha_{\max}} \partial^+ \nu(\alpha) \right) . \]

Proposition

The topological vector space \( S^\nu \) is not \( p \)-normable for any \( p > 0 \). Moreover,

- if \( p_0 < 1 \), then \( S^\nu \) is not \( p \)-locally convex for any \( p > p_0 \);
- if \( p_0 > 0 \), then \( S^\nu \) is \( p_0 \)-locally convex.
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- if \( p_0 > 0 \), then \( S^\nu \) is \( p_0 \)-locally convex.
Description of the topology

If $p_0 > 0$, the topology of $S^\nu$ is induced by the family of norms $\| \cdot \|_{b_{\infty,\infty}^{\alpha_{min} - \varepsilon}}$ together with the $p_0$-norms $\| \cdot \|_{\alpha,\varepsilon}$ defined by

$$\| x \|_{\alpha,\varepsilon} := \inf \left\{ \| x' \|_{b_{p_0,\infty}^s} + \| x'' \|_{b_{\infty,\infty}^\alpha} : x = x' + x'' \right\}$$

where $\alpha \in [\alpha_{min}, \alpha_{max}]$, $\varepsilon > 0$ and $s := \alpha + \frac{1-\nu(\alpha)}{p_0} - \varepsilon$. This family of $p_0$-norms may be made countable by taking a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ dense in $[\alpha_{min}, \alpha_{max}]$ and a sequence $\{\varepsilon_m\}_{m \in \mathbb{N}}$ of positive real numbers converging to 0.

Case $p_0 = 0$

If $\alpha_{min} > -\infty$, for every sequence $\{p_m\}_{m \in \mathbb{N}}$ of $]0, 1]$ converging to 0, the topology of the space $S^\nu$ can be defined by a sequence of $p_m$ semi-norms. Therefore, the space $S^\nu$ is locally pseudoconvex.
Description of the topology

If \( p_0 > 0 \), the topology of \( S^\nu \) is induced by the family of norms \( \| \cdot \|_{b_{\alpha_{min} - \varepsilon}^{\alpha_{min} - \varepsilon}} \) together with the \( p_0 \)-norms \( \| \cdot \|_{\alpha, \varepsilon} \) defined by

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Therefore, the space \( S^\nu \) is locally pseudoconvex.
More properties when $p_0 = 1$

- $S^\nu$ is a Frechet space
- For Frechet spaces, we have the following relations:

\[
\begin{array}{c}
\text{Nucléaire} \rightarrow \text{Schwartz} \\
\text{Montel} \rightarrow \text{Réflexif} \\
\text{Quasi-normable} \rightarrow \text{Condition de Densité} \\
\text{Distingué}
\end{array}
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**Proposition**

If $p_0 = 1$, the space $S^\nu$ is not nuclear but is a Schwartz space.
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If \( p_0 = 1 \), the space \( S^\nu \) is not nuclear but is a Schwartz space.
If $p_0 < 1$?

- The space $S^\nu$ is not locally convex
- Generalize the studied properties

**Définition**

A Hausdorff topological vector space $E$ is **Schwartz** if every 0-nbh $U$ contains a 0-nbh $V$ such that for every $\lambda > 0$, there exists a finite set $M \subset E$ such that

$$V \subset M + \lambda U.$$ 

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**Proposition**

The space $S^\nu$ is Schwartz.
Dual Space

If \( u \in (S^\nu)' \), then \( u \) can be identified with a sequence \( y \) such that

\[
u(x) = \sum_{j \in \mathbb{N}} \sum_{k=0}^{2^j-1} x_{j,k} y_{j,k}
\]

for every \( x \in S^\nu \). Indeed, we set

\[
y = \sum_{j \in \mathbb{N}} \sum_{k=0}^{2^j-1} u(\vec{e}^j,k) e^{j,k}.
\]

Notation:

\[
\|\beta\| = \begin{cases} 
-\infty & \text{if } \beta < 0 \\
\beta & \text{if } 0 \leq \beta \leq 1 \\
1 & \text{if } \beta \geq 1 
\end{cases}
\]

The dual profile of \( \nu \) is the function \( \nu' \) defined on \( \mathbb{R} \) by

\[
\nu' : \alpha' \mapsto \|\alpha' + \inf \{\alpha : \nu(\alpha) - \alpha > \alpha'\}\|.
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Dual Space

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Proposition
For every decreasing sequence \((\varepsilon_m)_{m \in \mathbb{N}}\) that converges to 0, the dual of \(S^\nu\) is
\[
(S^\nu)' = \bigcup_{\varepsilon > 0} S^{\nu_\varepsilon} = \bigcup_{m \in \mathbb{N}} S^{\nu_\varepsilon_m}
\]
where
\[
\nu_\varepsilon'(\alpha') := \nu'(\alpha' - \varepsilon) \; \forall \alpha' \in \mathbb{R}.
\]

Proposition
On the dual space, the strong topology and the inductive limit topology coincide.
Wavelet leaders

**Standard notation:** For \( j \in \mathbb{N}_0, k \in \{0, \ldots, 2^j - 1\} \),

\[
\lambda(j, k) := \{ x \in \mathbb{R} : 2^j x - k \in [0, 1] \} = \left[ \frac{k}{2^j}, \frac{k + 1}{2^j} \right],
\]

and for all \( j \in \mathbb{N}_0 \), \( \Lambda_j \) denote the set of all dyadic interval (of \([0, 1]\)) of length \( 2^{-j} \).

**Definition**

The wavelet leaders of a signal \( f \in L^2([0, 1]) \) are defined by

\[
d_\lambda := \sup_{\lambda' \subset \lambda} |c_{\lambda'}|, \quad \lambda \in \Lambda_j, \ j \in \mathbb{N}_0
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Definition

Let $\nu$ be an admissible profile. A sequence $c$ of $\Omega$ belongs to $\tilde{S}_\nu$ if for every $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$, there exists $J \in \mathbb{N}_0$ such that

$$\# \tilde{E}_j(C, \alpha)(c) \leq 2^{(\nu(\alpha) + \varepsilon)j} \quad \forall j \geq J,$$

where

$$\tilde{E}_j(C, \alpha)(c) := \left\{ k \in \{0, \ldots, 2^j - 1\} : d_{j,k} \geq C2^{-\alpha j} \right\}.$$

For every $c \in \Omega$, we define the asymptotic leader profile of $c$ by

$$\tilde{\nu}_c(\alpha) := \lim_{\varepsilon \to 0^+} \left( \limsup_{j \to +\infty} \left( \frac{\ln(\# \tilde{E}_j(1, \alpha + \varepsilon)(c))}{\ln(2^j)} \right) \right), \quad \alpha \in \mathbb{R}.$$

Proposition

The space $\tilde{S}_\nu$ is a linear space and

$$\tilde{S}_\nu = \{ c \in \Omega : \tilde{\nu}(\alpha) \leq \nu(\alpha) \quad \forall \alpha \in \mathbb{R} \}.$$
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