A brief survey on transitivity and Devaney’s chaos: autonomous and nonautonomous discrete dynamical systems

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Definition
Let $X$ be a metric space. If $f: X \to X$ is a continuous function, then $(X, f)$ is called a(n) (autonomous) discrete dynamical system.
Let $x \in X$. The orbit of $x$ is the sequence

$$x, f(x), f^2(x), \ldots, f^n(x), \ldots.$$
Introduction

**Definition (TT)**

A discrete dynamical system \((X, f)\) is said to be *topological transitive* if for every pair of nonempty open sets \(U\) and \(V\) in \(X\), there is a positive integer \(n\) such that \(f^n(U) \cap V \neq \emptyset\).
Definition (DO)

A discrete dynamical system \((X, f)\) is said to satisfy \emph{property (DO)} if there is a point \(x \in X\) such that the orbit of \(x\) is dense in \(X\).
(DO) does not imply (TT)

Take \( X = \{0\} \cup \{1/n\} \) and \( f : X \to X \) defined as \( f(1/n) = 1/(n + 1) \).
Neither \((TT)\) implies \((DO)\)

To this end take \(\mathbb{I}\) and the standard tent map
\(g(x) = 1 - |2x - 1|\) from \(\mathbb{I}\) into itself.
Let \(X\) be the set of all periodic points of \(g\) and \(f = g|_X\).
Then the system \((X, f)\) does not satisfy the condition \((DO)\), since \(X\) is infinite (dense in \(\mathbb{I}\)) while the orbit of any periodic point is finite. But the condition \((TT)\) is fulfilled.
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Theorem (Sylverman (1992))

If $X$ has no isolated point then $(DO)$ implies $(TT)$. If $X$ is separable and second category, then $(TT)$ implies $(DO)$. 
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Devaney’s chaos

A discrete dynamical system \((X, f)\) is called *Devaney chaotic* if the following conditions hold:

(i) \((X, f)\) is transitive;

(ii) the periodic points of \(f\) are dense in \(X\);

(iii) \(f\) has sensitive dependence on initial conditions.
Theorem (Banks et al. (1992))
Transitivity + $P(f)$ dense $\implies$ sensitive dependence on initial conditions.
Theorem (Alsedà et al. (1999))
If in the system $(X, f)$ the space $X$ is connected and has a disconnecting interval and $f$ is transitive, then $P(f)$ is dense in $X$. 
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Definition Let $X$ be a topological space, $f_n : X \to X$ a continuous function for each positive integer $n$, and $f_\infty = (f_1, f_2, \ldots, f_n, \ldots)$. The pair $(X, f_\infty)$ denotes the nonautonomous discrete dynamical system (NDS, for short) in which the orbit of a point $x \in X$ under $f_\infty$ is defined as the set

$$\text{orb}(x, f_\infty) = \{x, f_1(x), f_1^2(x), \ldots, f_1^n(x), \ldots\},$$

where

$$f_1^n := f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1,$$

for each positive integer $n$. 
Theorem

Suppose that $X$ is a second-countable space with the Baire property. If $(X, f_\infty)$ is transitive, then there exists a dense orbit.
Example

There is a NDS $(\mathbb{I}, g_\infty)$ which has a dense orbit but it is not transitive.
Example

There is a transitive NDS \((\mathbb{I}, g_\infty)\) with sensitive dependence on initial conditions such that the set of periodic points is not dense in \(\mathbb{I}\).
For Further Reading


For Further Reading


That’s all folks !!