A brief survey on transitivity and Devaney's chaos: autonomous and nonautonomous discrete dynamical systems

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Definition

Let X be a metric space. If $f\colon X \to X$ is a continuous function, then (X, f) is called a(n) (autonomous) discrete dynamical system.

Let $x \in X$.

The orbit of x is the sequence

$$x, f(x), f^2(x), \ldots\ldots, f^n(x), \ldots\ldots$$

Definition (TT)

A discrete dynamical system (X, f) is said to be *topological transitive* if for every pair of nonempty open sets U and V in X, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$.

Definition (DO)

A discrete dynamical system (X, f) is said to satisfy *property (DO)* if there is a point $x \in X$ such that the orbit of x is dense in X.

(DO) does not imply *(TT)*

Take $X = \{0\} \cup \{1/n\}$ and $f\colon X \to X$ defined as $f(1/n) = 1/(n+1)$.

Neither *(TT)* implies *(DO)*

To this end take $\mathbb{I}$ and the standard tent map $g(x) = 1 - |2x - 1|$ from $\mathbb{I}$ into itself.
Let X be the set of all periodic points of g and $f = g|_X$.
Then the system (X, f) does not satisfy the condition *(DO)*, since X is infinite (dense in $\mathbb{I}$) while the orbit of any periodic point is finite. But the condition *(TT)* is fulfilled.

Theorem (Sylverman (1992))

If X has no isolated point then *(DO)* implies *(TT)*. If X is separable and second category, then *(TT)* implies *(DO)*.

Devaney's chaos

A discrete dynamical system (X, f) is called *Devaney chaotic* if the following conditions hold:

- (i) (X, f) is transitive;
- (ii) the periodic points of f are dense in X;
- (iii) f has sensitive dependence on initial conditions.

Theorem (Banks et al. (1992))

Transitivity + $P(f)$ dense $\Longrightarrow$ sensitive dependence on initial conditions.

Theorem (Alsedà et al. (1999))

If in the system (X, f) the space X is connected and has a disconnecting interval and f is transitive, then $P(f)$ is dense in X.

Definition Let X be a topological space, $f_n\colon X \to X$ a continuous function for each positive integer n, and $f_\infty = (f_1, f_2, \ldots, f_n, \ldots)$.
The pair (X, f_∞) denotes the *nonautonomous discrete dynamical system* (NDS, for short) in which the *orbit of a point* $x \in X$ under f_∞ is defined as the set

$$\mathrm{orb}(x, f_\infty) = \{x, f_1(x), f_1^2(x), \ldots, f_1^n(x), \ldots\},$$

where

$$f_1^n := f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1,$$

for each positive integer n.

Theorem

Suppose that X is a second-countable space with the Baire property. If (X, f_∞) is transitive, then there exists a dense orbit.

Example

There is a NDS $(\mathbb{I}, g_\infty)$ which has a dense orbit but it is not transitive.

Example

There is a transitive NDS $(\mathbb{I}, g_\infty)$ with sensitive dependence on initial conditions such that the set of periodic points is not dense in $\mathbb{I}$.

For Further Reading

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That's all folks !!