# Real analytic functions: spaces, operators and ideals

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Two more descriptions:

$$A(\mathbb{R}^d) = \bigcup \{ H(\Omega) \,:\, \Omega \text{ open } \mathbb{R}^d \subset \Omega \subset \mathbb{C}^d \}$$

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$$A(\mathbb{R}^d) = \lim\operatorname{proj}_k H(B_k)$$

where $B_k = \{x \in \mathbb{R}^d \,:\, |x| \leq k\}$. Set $U_r = \{x \in \mathbb{C}^d \,:\, |x| < r\}$.

The space $H(\Omega)$ with the compact-open topology is a nuclear Fréchet space.

$H(B_k) = \lim\operatorname{ind}_l H^\infty(B_k + U_{1/l})$ is a nuclear (LB)-space, i. e. inductive limit of a sequence of Banach spaces with nuclear linking maps.

There are two ways of introducing a topology on $A(\mathbb{R}^d)$:
1. inductive topology of the $H(\Omega)$
2. projective topology of the $H(B_k)$

1. $\rightsquigarrow$ ultrabornological topology
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is continuous for all $\Omega$ and $k$. And it is the unique 'reasonable' topology which gives the classical convergence of sequences in $A(\mathbb{R}^d)$:

$f_n \longrightarrow f \Leftrightarrow f_n \longrightarrow f$ uniformly on a complex neighborhood of every $x \in \mathbb{R}^d$.

## Structural results

$E$ Fréchet space

**Theorem** (Domański-Langenbruch): *$E$ is isomorphic to a subspace of $A(\mathbb{R}^d)$ if and only if $E$ is $n^{1/d}$-nuclear and has property* $(\underline{\mathrm{DN}})$*.*

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A Fréchet space with a fundamental sequence of seminorms $(\|\cdot\|_n)$ defining the topology is said to have property ($\underline{DN}$) if

$$\exists\ n\ \ \forall\ k\ \ \exists\ l, C > 0, \tau \in ]0,1[ \qquad \|x\|_k \leq C\|x\|_n^{\tau}\|x\|_l^{1-\tau}$$

for every $x \in E$.

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*nuclear* would be:
$\ldots\ldots$ the ratios of the semiaxes are summable.

**Theorem** (Domański-Frerick-V.): *$E$ is isomorphic to a quotient space of $A(\mathbb{R}^d)$ if and only if $E$ is is $n^{1/d}$-nuclear and has property $(\overline{\overline{\Omega}})$.*

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$$\forall\ k\ \ \exists\ m\ \ \forall\ n, \vartheta \in ]0,1[\ \ \exists\ C\ \ \forall\ u \in E' \qquad \|u\|_m^* \leq C\ {\|u\|_k^*}^{\vartheta} {\|u\|_n^*}^{1-\vartheta},$$

Here $\|\cdot\|^*$ denotes the dual norm for $\|\cdot\|$.

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*Remark.* A Fréchet space $E$ has $(\overline{\overline{\Omega}})$ if and only if

$$\forall\ V\ \ \exists\ U\ \ \forall\ W, \gamma > 0\ \ \exists\ C\ \ \forall\ r > 0 \qquad U \subseteq C\left(\frac{1}{r^\gamma} V + rW\right),$$

where $U$, $V$, $W$ are 0-neighbourhoods in $E$.

REMARK: $(\overline{\overline{\Omega}})$ is very restrictive, Fréchet spaces in analysis usually don't have $(\overline{\overline{\Omega}})$.

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Theorem holds in much more general context:
Let $X$ be a $d$-dimensional, connected and $\sigma$-compact real analytic manifold.

Let $(E, \pi, X)$ be a $p$-dimensional real analytic vector bundle with base $X$.

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Explanation: $E$ is a real analytic manifold, $\pi : E \longrightarrow X$ a surjective real analytic map. There is an open covering $U_\iota$, $\iota \in I$ and for every $\iota$ a bi-analytic map $\chi_\iota : \pi^{-1}(U_\iota) \longrightarrow U_\iota \times \mathbb{R}^p$ such that $\pi_1 \circ \chi_\iota = \pi$ where $\pi_1(x, \xi) = x$ and $\pi_2 \circ \chi_\iota : \pi^{-1}(x) \longrightarrow \mathbb{R}^p$ is linear.

Examples: $X \times \mathbb{R}^p$ ("trivial bundle"), tangent-bundle, cotangent-bundle

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For open $\omega \subset X$ a section on $\omega$ is a map $s : \omega \longrightarrow E$ with $\pi \circ s = \mathrm{id}$. By $A_E(\omega)$ we denote the linear space of real analytic sections on $\omega$. Since $A_E(U_\iota) \cong A(U_\iota)^p$ the $A_E(\omega)$ define a locally free sheaf $\mathscr{A}_E$ on $X$.

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**Application:**

Let $(E_j, \pi_j, X)$ be $q_j$-dimensional real analytic vector bundles, $j = 0, 1$.

Let $\mathscr{E}_{E_0}(X)$ and $\mathscr{D}'_{E_1}(X)$ denote the $C^\infty$- resp. distributional sections on $X$ and let $L$ denote a differential operator acting from $\mathscr{E}_{E_0}(X)$ to $\mathscr{D}'_{E_1}(X)$. We call it elliptic if $\ker L \subset A_{E_0}(X)$.

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Consequence: If $\ker L$ is infinite dimensional then it is not complemented and $L$ has no solution operator.

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## Interpolational invariants on $A(\mathbb{R}^d)$

**Theorem** (Bonet-Domański): *$A(\mathbb{R}^d)$ satisfies the dual interpolation estimate for big $\vartheta$ (which implies $(\overline{\overline{\Omega}})$ on Fréchet quotients):*

$$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m, \vartheta \in ]\vartheta_0, 1[\ \exists\ k, C\ \forall\ y \in H(B_N)' :$$
$$\|y\|^*_{M,m} \leq C \left( {\|y\|^*_{K,k}}^{1-\vartheta} \cdot {\|y\|^*_{N,n}}^{\vartheta} \right)$$

## Homological concepts

$\mathcal{X} : X_1 \overset{j_1^2}{\longleftarrow} X_2 \overset{j_2^3}{\longleftarrow} \dots$ projective spectrum. Then we have $X := \mathrm{Proj}^0\mathcal{X} = \lim\mathrm{proj}\,\mathcal{X}$ and its first derived functor $\mathrm{Proj}^1\mathcal{X}$.

With $\sigma(x_1, x_2, \dots) = (x_1 - j_1^2 x_2, x_2 - j_2^3 x_3, \dots)$ exact sequence

$$0 \longrightarrow X \longrightarrow \prod_j X_j \overset{\sigma}{\longrightarrow} \prod_j X_j \longrightarrow \mathrm{Proj}^1\mathcal{X} \longrightarrow 0.$$

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$\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ reduced spectra defining (PLB)-spaces $X$, $Y$, $Z$.

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z} \longrightarrow 0$$

exact sequence of spectra, which means a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
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 & & \uparrow & & \uparrow & & \uparrow & & \\
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**Theorem:** $\mathrm{Proj}^1 A(\mathbb{R}^d) = 0.$

**Application:** $Y = Z = A(\mathbb{R}^d)$, $P$ polynomial of degree $m$, map $P(D) : A(\mathbb{R}^d) \longrightarrow A(\mathbb{R}^d)$
$P(D) : H(B_r) \longrightarrow H(B_r)$ is surjective for all $r > 0$
gives for every $n$ an exact sequence

$$0 \longrightarrow N_P(B_n) \longrightarrow H(B_n) \stackrel{P(D)}{\longrightarrow} H(B_n) \longrightarrow 0$$

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$$0 \longrightarrow N_P(\mathbb{R}^d) \longrightarrow A(\mathbb{R}^d) \xrightarrow{P(D)} A(\mathbb{R}^d) \longrightarrow \operatorname{Proj}^1 N_P(\mathbb{R}^d) \longrightarrow 0.$$

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**Interpretation:** Assume that we are given a set $V$, a linear space $A$ of functions on $V$, and a family $(b_{k,n})_{k,n}$ of weight functions on $V$ such that

$$b_{k,n} \leq b_{k+1,n} \quad \text{and} \quad b_{k,n} \geq b_{k,n+1} \quad \text{for } k, n \in \mathbb{N}.$$

We assume

$$X'_{k,n} = \{f \in A \mid \|f\|^*_{k,n} = \sup_{t \in V} |f(t)| \exp(-b_{k,n}(t)) < \infty\}$$

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Then condition 2. becomes ("abstract Phragmén-Lindelöf condition"):
$\forall\ \mu\ \exists\ n,\ k\ \forall\ K,\ m\ \exists\ N,\ S\ \forall\ f \in X'_\mu : (a) \wedge (b) \Rightarrow (c)$, where
(a) $\log|f| \leq b_{\mu,n}$,
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In our case by the Ehrenpreis-Palamodov fundamental principle:
$V = \{z \in \mathbb{C}^d\ :\ P(z) = 0\}$, $A = H(V)$, $b_{k,n} = \frac{1}{n}|x| + k|y|$.

Therefore (Hörmander):

$P(D) : A(\mathbb{R}^d) \longrightarrow A(\mathbb{R}^d)$ surjective $\Leftrightarrow$ Concrete Phragmén-Lindelöf condition.

## The local Phragmén-Lindelöf condition

NOTATION: $V_a$ germ of an analytic set at $a \in \mathbb{R}^d$, $X_a = V_a \cap \mathbb{R}^d$. We identify $V_a$ with a bounded representative. For a function $f$ on a set $M$ we set $\|f\|_M = \sup_{x \in M} |f(x)|$.

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**Definition** (HÖRMANDER): *$V_a$ satifies $PL_{loc}$ if there is a constant $A > 0$ and a neighborhood $W_a$ of $a$ in $V_a$, such that for every plurisubharmonic function $u$ on $V_a$ with $u \leq 1$ on $V_a$ and $u \leq 0$ on $X_a$ we have: $u(z) \leq A|\operatorname{Im} z|$ for all $z \in W_a$.*

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**Equivalent definition:** *...for every holomorphic function $f$ on $V_a$ and all $z \in W_a$ we have:*

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**Definition:** A complex analytic set in a neighborhood of $\mathbb{R}^d$ is of type $PL$, if it satisfies $PL_{loc}$ in all of its real points.

## Global solutions for differential operators

**Theorem** (Hörmander): $P(D) : A(\mathbb{R}^d) \longrightarrow A(\mathbb{R}^d)$ *is surjective if and only if* $V_{P_m}$ *satisfies* $PL_{loc}$ *in every* $x_0 \in V(P_m) \cap \mathbb{R}^d$, $|x_0| = 1$.

Here $P_m$ is the principal part of $P$.

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Remark: Condition depends only on the principal part!

Examples: 1. All irreducible factors of $P_m$ elliptic or hyperbolic $\rightsquigarrow +$
2. $P_m(x) = x_1^m + \cdots + x_d^m \rightsquigarrow +$
3. $d \geq 3$ (space dimension $\geq 2$) heat equation, Schrödinger equation $\rightsquigarrow$ - (original example of di Giorgio, Cattabriga, Piccinini)
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Reason for Case 4.: We may assume $P_m(x) = Ax_1^m + \ldots$ then

$$P_m(x) = A \prod_1^m (x_1 - a_j x_2)$$

$a_j \in \mathbb{R} \Rightarrow D_1 - a_j D_2$ hyperbolic and clearly surjective!
$a_j \in \mathbb{R} \Rightarrow D_1 - a_j D_2$ elliptic, hence surjective.

## Real analytic parameter dependence

For $P \in \mathbb{C}[z_1, \ldots, z_n]$ we set $P^+ = P$, considered as a polynomial in $\mathbb{C}[z_1, \ldots, z_{n+1}]$. For open $\Omega \subset \mathbb{R}^d$ problem:
When is $P^+(D) : A(\Omega \times \mathbb{R}) \longrightarrow A(\Omega \times \mathbb{R})$ surjective?

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# Right inverses

**Theorem:** *Let* $n > 1$ *and* $\Omega \subset \mathbb{R}^d$ *open. If* $P(D) : A(\Omega) \longrightarrow A(\Omega)$ *admits a right inverse, then* $P_m$ *has no elliptic factor.*

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**Theorem:** *The Cauchy problem*

$$P(D)f = 0, \quad \frac{\partial^k}{\partial x_1^k} f(0, x') = f_k(x'), \quad k = 0, \ldots, m-1$$

*is uniquely solvable for all* $f_0, \ldots, f_{m-1} \in A(\mathbb{R}^{d-1})$ *with* $f \in A(\mathbb{R}^d)$*, if and only if* $P_m(D)$ *is hyperbolic with respect to* $e_1$*.*

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CASE OF $d = 2$:

**Theorem:** *For $d = 2$ the following are equivalent:*
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# Analysis of problem

**Theorem** (Momm): *For any* $P \neq 0$ *and* $r > 0$ *the map* $P(D) : H(B_r) \longrightarrow H(B_r)$ *has a continuous linear right inverse.*

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Can replace $\Rightarrow$ with $\Leftrightarrow$ if $\mathrm{Proj}^1(L(A(\mathbb{R}^d), H(B_n)))_n = 0$. **Unknown!**

## Real analytic subvarieties of $\mathbb{R}^d$

**Definition:** A real analytic subvariety of $\mathbb{R}^d$ is a closed subset $X \subset \mathbb{R}^d$, which is locally common zero set of finitely many real analytic functions.

**Definition:** A real analytic function on $X$ is a continuous function on $X$ which is in a neighborhood of every $a \in X$ restriction of a real analytic function on a neighborhood of a in $\mathbb{R}^d$.

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**Problem:** 1. When is $J_X(\mathbb{R}^d)$ complemented in $A(\mathbb{R}^d)$?
2. When does there exist a continuous linear extension operator $A(X) \longrightarrow A(\mathbb{R}^d)$?

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**Definition:** A real analytic subvariety of $\mathbb{R}^d$ is a closed subset $X \subset \mathbb{R}^d$, which is locally common zero set of finitely many real analytic functions.

**Definition:** A real analytic function on $X$ is a continuous function on $X$ which is in a neighborhood of every $a \in X$ restriction of a real analytic function on a neighborhood of a in $\mathbb{R}^d$.

NOTATION: $J_X(\mathbb{R}^d) = \{f \in A(\mathbb{R}^d) \,:\, f|_X = 0\}$, the ideal of $X$.

**Problem:** 1. When is $J_X(\mathbb{R}^d)$ complemented in $A(\mathbb{R}^d)$?
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**Pathologies:**
1.For $X = \{(x, y, z) \in \mathbb{R}^3 \,:\, z(x^2 + y^2) = x^3\}$ ("Cartan's umbrella") not every $f \in A(X)$ can be extended to the whole of $\mathbb{R}^d$, not even to a neighborhood of $X$, e.g. $f(x, y, z) = x/(x^2 + y^2 + (z - 1)^2)$.

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2. Can happen $J_X(\mathbb{R}^d) = \{0\}$, e.g. $X = \{(x,y,z) \in \mathbb{R}^3 \,:\, z(x^2+y^2) = x^3 a(z)\}$, $a(z) = \exp(1/(z^2-1)$ for $z^2 < 1$, zero otherwise.

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**Lemma** (Bruhat-Whitney): *The common zero set of arbitrary many functions in $A(\mathbb{R}^d)$ is $\mathbb{C}$-analytic.*

Consequences: 1. $X$ is $\mathbb{C}$-analytic $\Leftrightarrow X = \mathrm{loc} J_X(\mathbb{R}^d)$.
2. If you solve problem 1. in terms of conditions on $X$ for $\mathbb{C}$-analytic $X$, then you solve problem 1. in terms of $\hat{X} := \mathrm{loc} J_X(\mathbb{R}^d)$ for any $X$.

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**Lemma** (Bruhat-Whitney): *If $X$ is $\mathbb{C}$-analytic then it admits a global complexification.*

Global complexification = A complex variety $V_x$ in a neighborhood of $X$ in $\mathbb{C}^d$ such that
1. $X = V_X \cap \mathbb{R}^d$
2. every $f \in H(V_X)$ which vanishes on $X$ vanishes also on $V_X$.

# Coherent varieties

Remark: Cartan's umbrella is $\mathbb{C}$-analytic, its global complexification is the complex zero set of the polynomial, but the local complexification in $(0, 0, z)$, $z \neq 0$ is not the germ of the global complexification.

Notation: For $a \in X$: $J_X(a)$ is the ideal of $X$ in $\mathscr{O}_a$
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$X$ *coherent* : for every $a \in X$ generators of $J_X(a)$ generate $J_X(b)$ for $b$ in some neighborhood of $a$. Remark: $J_X(a)$ is finitely generated!

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Example: Cartan's umbrella is $\mathbb{C}$-analytic, but not coherent!
$J_X(0) = P \cdot \mathscr{O}_0$, but $J_X(0,0,z) = x \cdot \mathscr{O}_{(0,0,z)} + y \cdot \mathscr{O}_{(0,0,z)}$ for any $z \neq 0$.

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$X$ coherent $\Rightarrow$ none of the pathologies can happen, in particular $X$ is $\mathbb{C}$-analytic and there is an exact sequence

$$0 \longrightarrow J_X(\mathbb{R}^d) \longrightarrow A(\mathbb{R}^d) \longrightarrow A(X) \longrightarrow 0.$$

If $X$ is coherent, then Problems 1. and 2. are equivalent.

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*1. $(P)$ is complemented in $A(\mathbb{R}^d)$.*
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*3. There is a continuous linear operator $S = S_P \in L(A'(\mathbb{R}^d))$ so that $P \cdot S(\mu) = \mu$ for all $\mu \in A'(\mathbb{R}^d)$.*

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Notation: For any complex subvariety $V$ of a neighborhood $\Omega \subset \mathbb{C}^d$ we set

$$\begin{aligned} J_V(\mathbb{R}^d) &= \{f \in A(\mathbb{R}^d) \, : \, F|_V = 0\} \\ H_V(X) &= \{\text{germs around } X \text{ of holomorphic functions on } V\} \end{aligned}$$

where $F$ is the germ of an extension of $F$ to $\mathbb{C}^d$ and $X = V \cap \mathbb{R}^d$. Gives exact sequence

$$0 \longrightarrow J_V(\mathbb{R}^d) \longrightarrow A(\mathbb{R}^d) \longrightarrow H_V(X) \longrightarrow 0.$$

**Lemma:** *1. $P$ irreducible $\Rightarrow (P) = J_{V_P}(\mathbb{R}^d)$.*
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CONSEQUENCE: In both cases the ideal is complemented $\Leftrightarrow$ the respective exact sequence splits.

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**Theorem** (DOMAŃSKI-V.): *If the exact sequence*

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**Lemma:** *If $V$ satisfies $PL_{loc}$ in $a \in X$ then $V_a$ is the complexification of $X_a$, it satisfies $PL_{loc}$ in an neighborhood of $a$ and $X_a$ is coherent.*

**Corollary:** *If the exact sequence in the theorem splits then $J_V(\mathbb{R}^d) = J_X(\mathbb{R}^d)$ and $H_V(X) = A(X)$, in particular, the exact sequence*

$$O \longrightarrow J_X(\mathbb{R}^d) \longrightarrow A(\mathbb{R}^d) \longrightarrow A(X) \longrightarrow 0$$

*splits.*

## Varieties of type $PL$

Let $V_a$ be the germ of a complex variety in the real point $a$, $X_a = V_a \cap \mathbb{R}^d$.

$$\omega_{a,V}(z) = \limsup_{\zeta \to z} \sup\{u(\zeta) \ : \ \text{u plurisubharmonic on } V_a,\ u \le 1, u \le 0 \text{ on } X_a\}.$$

**Definition:** $V_a$ satisfies $PL_{loc}$ if $\omega_{a,V} \prec |\mathrm{Im}\, z|$.

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Let $X_a$ be the germ of a real analytic variety in the real point $a$. Let $\widehat{X}_a$ be the (local) complexification of $X_a$.

**Definition:** The germ $X_a$ of a real analytic variety in the real point $a$ is of type $PL$ if $\widehat{X}_a$ satisfies $PL_{loc}$. $X$ is of type $PL$ if it is of type $PL$ in every point.

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1. The "non-$PL$ points" are a closed subset of the set of singular points of $X$.
"non-$PL$" = special type of singularity.
2. Type $PL \Rightarrow$ coherence.
3. If $V_a$ satisfies $PL_{loc}$ then so does each of its irreducible branches.

Hörmander '73, Meise-Taylor-V. '90, Braun-Meise-Taylor '04 ff., V. '07 f.

## Discussion of the conditions

$P$ polynomial, $V = V_P$, $X = V \cap \mathbb{R}^d$.

CONDITIONS:

1. $V_a$ satisfies $PL_{loc}$ in every point of $X \leftrightsquigarrow (P)$ complemented in $A(\mathbb{R}^d)$
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EXAMPLES:
1. $P(x, y) = x^2 + y^2$ : $V_P$ purely 1-dimensional, $X = \{0\} \Rightarrow V_P$ is not the complexification of $X \Rightarrow (P)$ is not complemented. Clearly $J_X(\mathbb{R}^d)$ is complemented.

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2. $P(x,y) = y^3 - x^3(1+x^2)$ irreducible, $X$ is of pure dimension 1. In $\mathscr{O}_0$ it decomposes into one real and two complex factors $\Rightarrow$
$V_P$ not $PL_{loc}$ in 0 $\Rightarrow (P)$ not complemented.

In $A(\mathbb{R}^d)$it decomposes into two real valued factors:

$$f_1(x,y) = y - x\sqrt[3]{1+x^2}, \quad f_2 = \left(y + \tfrac{x}{2}\sqrt[3]{1+x^2}\right)^2 + \tfrac{3}{4}x^2\left(1+x^2\right)^{\frac{2}{3}}.$$

$f_1$ describes $X$, $f_2$ vanishes in 0 only. There is a continuous linear projection in $A(\mathbb{R}^d)$onto $J_X(\mathbb{R}^d)$, namely $f(x,y) \mapsto f(x,y) - f(x, x\sqrt[3]{1+x^2})$.

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**Proposition:** *In the following cases $PL$ implies that $J_X(\mathbb{R}^d)$ is complemented: $X$ compact, $X$ homogeneous, $X$ analytic cover with compact critical set.*

**Theorem** (Domański-V.): *If the $\mathbb{C}$-analytic variety $X$ is compact or homogeneous, then the following are equivalent:*
*1. $J_X(\mathbb{R}^d)$ is complemented in $A(\mathbb{R}^d)$.*
*2. There exists a continuous linear extension operator $A(X) \longrightarrow A(\mathbb{R}^d)$.*
*3. $X$ is of type $PL$.*

Remark: $X$ homogeneous, then: $X$ of type $PL \Leftrightarrow X_0$ of type $PL$.

## Algebraic curves

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Examples:
1. $y^2 = x^3$ (Neill Parabola) $\rightsquigarrow$ bad
2. $y^2 = x^2(x+1)$ (Newton's Knot) $\rightsquigarrow$ good
3. $x^3 + y^3 - 3xy = 0$ (Cartesian Leaf) $\rightsquigarrow$ good
4. $3(x^2+y^2)^2 + 8x(3y^2 - x^2) + 6(x^2+y^2) = 1$ (Deltoid) $\rightsquigarrow$ bad

## Analysis of extension problem

**Theorem:** *If $X$ is of type PL, then for every compact $K \subset \mathbb{R}^d$ there is a continuous linear map $\varphi_K : A(X) \longrightarrow H(K)$ so that $\varphi_K f|_{K \cap X} = f|_{K \cap X}$ for all $f \in A(X)$.*

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Get exact sequence of spectra

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$$\begin{aligned} 0 \longrightarrow L(A(X), J_X(\mathbb{R}^d) \longrightarrow L(A(X), A(\mathbb{R}^d)) \xrightarrow{\rho} L(A(X), A(X)) \longrightarrow \\ \mathrm{Proj}^1(L(A(X), J_X(B_n)))_n \longrightarrow \mathrm{Proj}^1(L(A(X), H(B_n)))_n \longrightarrow \end{aligned}$$

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$\mathrm{Proj}^1(L(A(X), J_X(B_n)))_n = 0 \Rightarrow \rho$ surjective $\Leftrightarrow$ exists extension operator $A(X) \longrightarrow A(\mathbb{R}^d)$.

Can replace $\Rightarrow$ with $\Leftrightarrow$ if $\mathrm{Proj}^1(L(A(X), H(B_n)))_n = 0$. **Unknown!**

## Local ideals

$X \in \mathbb{R}^d$ germ of a real analytic set in 0, $J_X \subset \mathscr{O}$ its ideal and $V = \text{ loc } J_X \subset \mathbb{C}^d$. Then $J_X = J_V$. $V$ is the *local complexification* of $X$.

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**Proposition:** *The sequence splits.*

## Local ideals

$X \in \mathbb{R}^d$ germ of a real analytic set in 0, $J_X \subset \mathscr{O}$ its ideal and $V = \text{ loc } J_X \subset \mathbb{C}^d$. Then $J_X = J_V$. $V$ is the *local complexification* of $X$.

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The complex sequence is 'graded exact', the real in general not. Example: Cartan's umbrella.
$X$coherent $\Rightarrow$ real sequence is 'graded exact'.

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Application: $\mathcal{A}^E$, $\mathcal{A}_X^E$ sheaves of $E$-valued real analytic functions on $\mathbb{R}^d$, resp. $X$, $\mathcal{J}_X^E$ $E$-valued “ideal”-sheaf. Then:

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**Theorem:** *If* $X$ *is of type* $PL$ *then for every Fréchet space* $E$ *the map* $\rho : A(\mathbb{R}^d, E) \longrightarrow A(X, E)$ *is surjective. If* $X$ *is algebraic, then the converse holds.*

## Fréchet valued real analytic functions

Let $E$ be a Fréchet space (or a complete locally convex space). An $E$-valued function $f$ on $\mathbb{R}^d$ is called real analytic if $y \circ f$ is real analytic for every $y \in E'$.

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**Problem:** Let $E$ be a Fréchet space. When can every $E$-valued real analytic function on $X$ be extended to $\mathbb{R}^d$?

EXAMPLE: $X$ compact, coherent $\Rightarrow A(X)' = H(X)'$ is a Fréchet space
If $f : x \mapsto \delta_x$ can be extended to $\mathbb{R}^d$, then the ideal $J_X(\mathbb{R}^d)$ is complemented in $A(\mathbb{R}^d)$.

**Remark:** $A(\mathbb{R}^d, E) = A(\mathbb{R}^d)\hat{\otimes}_\pi E$. Our problem: exactness of
$0 \longrightarrow J_X(\mathbb{R}^d)\hat{\otimes}_\pi E \longrightarrow A(\mathbb{R}^d)\hat{\otimes}_\pi E \longrightarrow A(X)\hat{\otimes}_\pi E \longrightarrow 0.$

## First approach

Let $E$ be a Fréchet space, $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \ldots$ a fundamental system of seminorms and $E_k = (E/\ker\|\cdot\|_k, \|\cdot\|_k)\hat{}$ the local Banach spaces.

Then

$$A(\mathbb{R}^d, E) = \lim\operatorname{proj}_k H(B_k, E_k) = \lim\operatorname{proj}_k \lim\operatorname{ind}_l H^\infty(B_k + iU_{1/l}, E_k).$$

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Recall: $\mathcal{X} : X_1 \leftarrow X_2 \leftarrow \dots$ projective spectrum. Then we have $X := \operatorname{Proj}^0\mathcal{X} = \lim\operatorname{proj}\mathcal{X}$ and its first derived functor $\operatorname{Proj}^1\mathcal{X}$. In the case of a reduced spectrum of (LB)-spaces $\operatorname{Proj}^1\mathcal{X} =: \operatorname{Proj}^1 X$ depends only on $X$.

We obtain “long exact sequence”

$$0 \longrightarrow J_X(\mathbb{R}^d, E) \longrightarrow A(\mathbb{R}^d, E) \longrightarrow A(X, E) \longrightarrow \mathrm{Proj}^1 J_X(\mathbb{R}^d, E) \longrightarrow \dots$$

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**Theorem:** $\mathrm{Proj}^1 J_X(\mathbb{R}^d, E) = 0 \Leftrightarrow E \in (\overline{\overline{\Omega}})$.

Example: $X$ compact coherent, then $E = A(X)' \in (\overline{\overline{\Omega}}) \Leftrightarrow X$ consists of finitely many discrete points.

## Second approach

We consider the sequence of spectra

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J_X(B_1, E) & \longrightarrow & A(B_1, E) & \longrightarrow & A(X \cap B_1, E) & \longrightarrow & 0 \\
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**Theorem:** *For every Fréchet space and $X$ we have $H^1(\mathscr{J}_X^E,\mathbb{R}^d)=0$.*

**Corollary:** *If the sequence of spectra is exact, then every $E$-valued real analytic function on $X$ can be extended to $\mathbb{R}^d$.*

## Sketch of proof of the theorem

Let $\Omega \subset \mathbb{C}^d$ be an open holomorphically convex neighborhood of $\mathbb{R}^d$.

**Lemma:** *There is a compact exhaustion $K^r$, $r > 0$ of $\Omega$ such that for every $r > 0$ and compact $K \subset \Omega$ there are continuous linear maps*

$$A : J_X(B_r) \longrightarrow J_X(\mathbb{R}^d) \text{ and } B : J_X(B_r) \longrightarrow J_V(K^r)$$

*so that for any $f \in J_X(B_r)$ we have*

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Remark: There is $\Omega$ and some complex subvariety $V \subset \Omega$ such that $X = V \cap \mathbb{R}^d$.

**Proposition:** *For any complete locally convex space the restriction map $H^1(\Omega, \mathscr{J}_V^E) \longrightarrow H^1(\mathbb{R}^d, \mathscr{J}_X^E)$ is surjective.*

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Cartan-Oka+Grothendieck $\Rightarrow$ For Fréchet space $E$: $H^1(\Omega, \mathscr{J}_V^E) = 0$.

## Main Theorem

**Lemma:** *If $X$ is of type $PL$, then for every compact $K \subset \mathbb{R}^d$ there is a continuous linear map $\varphi_K : A(X) \longrightarrow H(K)$ so that $\varphi_K f|_{K \cap X} = f|_{K \cap X}$ for all $f \in A(X)$.*

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**Theorem:** *Let $X$ be a real analytic subvariety of $\mathbb{R}^d$ of type $PL$ then for every Fréchet space $E$ the restriction map $A(\mathbb{R}^d, E) \longrightarrow A(X, E)$ is surjective, i.e. every $E$-valued real analytic function on $X$ can be extended to an $E$-valued real analytic function on $\mathbb{R}^d$.*

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**Definition** $X$ is called an algebraic subvariety of $\mathbb{R}^d$ if it is the common zero set of finitely many polynomials.

**Theorem:** *For an algebraic subvariety $X$ of $\mathbb{R}^d$ the following are equivalent:*

1. *For every Fréchet space $E$ the restriction map $A(\mathbb{R}^d, E) \longrightarrow A(X, E)$ is surjective.*
2. *$X$ is of type $PL$.*