Projections in duals to Asplund spaces made without Simons' lemma

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Necessity to play a "volleyball" between X and X^*:

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1° V is separable.

2° V is non-separable. Make a suitable separable reduction and apply 1°. □

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(Everything is standard, no extra ideas of Ch. Stegall) □

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References

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- M. Cúth, O. Kalenda, *Rich families and elementary submodels*, http://arxiv.org/abs/1308.1818
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References

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- Ch. Stegall, *Spaces of Lipschitz functions on Banach spaces*, Functional Analysis Essen 1991, 265–278; Lect. Notes in Pure and Appl. Math. **150** New York, Dekker 1994.

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THANK YOU FOR YOUR ATTENTION