Integral Equations for Computational Electromagnetics

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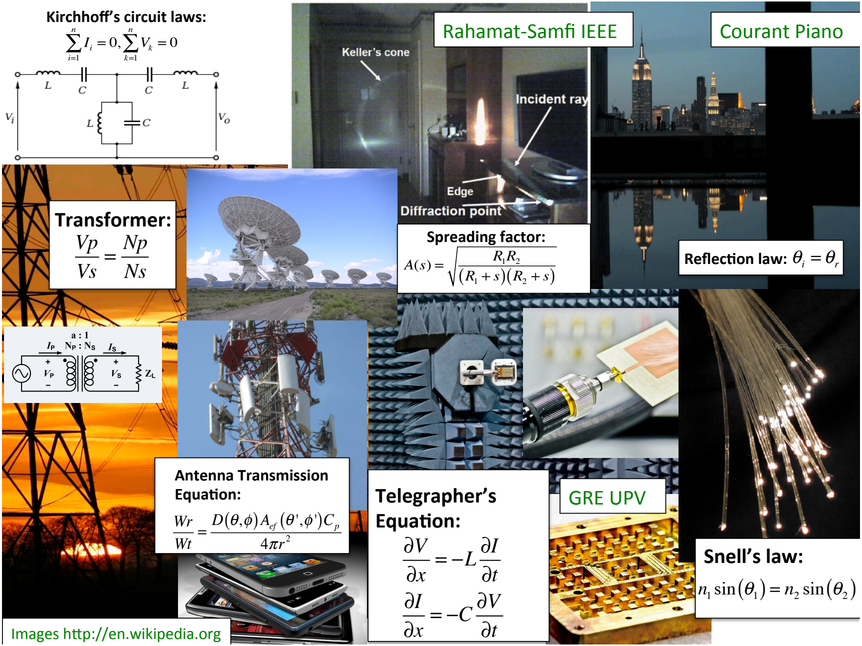
Maxwell's Equations

Maxwell's Equations:

$$
\begin{aligned}
\nabla \cdot \mathcal{E}(\mathbf{x}, t) &= \frac{\rho}{\epsilon} && \text{Gauss's law} \\
\nabla \times \mathcal{E}(\mathbf{x}, t) + \mu \frac{\partial \mathcal{H}}{\partial t}(\mathbf{x}, t) &= 0 && \text{Faraday's law} \\
\nabla \cdot \mathcal{H}(\mathbf{x}, t) &= 0 && \text{Magnetic Gauss's law} \\
\nabla \times \mathcal{H}(\mathbf{x}, t) - \epsilon \frac{\partial \mathcal{E}}{\partial t}(\mathbf{x}, t) &= \mathbf{J} && \text{Amper's law}
\end{aligned}
$$

$\mathcal{E}, \mathcal{H}$ electric and magnetic fields.



Kirchhoff's circuit laws:
$\sum_{i=1}^{n} I_i = 0, \sum_{k=1}^{n} V_k = 0$
V_i L C C L V_o L C
Rahamat-Samfi IEEE
Keller's cone
Incident ray
Edge
Diffraction point
Courant Piano
Transformer:
$\frac{Vp}{Vs} = \frac{Np}{Ns}$
a : 1
I_P $N_P : N_S$ I_S
V_P V_S Z_L
Spreading factor:
$A(s) = \sqrt{\frac{R_1 R_2}{(R_1+s)(R_2+s)}}$
Reflection law: $\theta_i = \theta_r$
Antenna Transmission Equation:
$\frac{Wr}{Wt} = \frac{D(\theta,\phi) A_{ef}(\theta',\phi') C_p}{4\pi r^2}$
Telegrapher's Equation:
$\frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t}$
$\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t}$
GRE UPV
Snell's law:
$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$
Images http://en.wikipedia.org

Electrostatics

Electrostatics:

$$\nabla \cdot \mathcal{E}(\mathbf{x}) = \rho$$
$$\nabla \times \mathcal{E}(\mathbf{x}) = 0$$

Electrostatic Potential:

$$\mathcal{E}(\mathbf{x}) = -\nabla\phi(\mathbf{x})$$

$$\Delta\phi(\mathbf{x}) = -\rho$$

Problem

Let $\rho \in C_0(\mathbb{R}^3) \cap C^{0,\alpha}(\mathbb{R}^3)$, *find* $\phi(\mathbf{x}) \in C^2(\mathbb{R}^3)$ *that verifies:*

$$\Delta\phi(\mathbf{x}) = -\rho$$

and the radiation condition: $\phi(\mathbf{x}) = o(1)$ *unif. in all directions* $\frac{\mathbf{x}}{|\mathbf{x}|}$

Solution: (convolution with the Green's function $g = \frac{1}{4\pi|\mathbf{x}|}$)

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\rho(\mathbf{y})dV_{\mathbf{y}} = g * \rho = V[\rho]$$

Electrostatics 2D

Problem

Let $\rho \in C_0(\mathbb{R}^2) \cap C^{0,\alpha}(\mathbb{R}^2)$, *find* $\phi(\mathbf{x}) \in C^2(\mathbb{R}^2)$ *that verifies:*

$$\Delta\phi(\mathbf{x}) = -\rho$$

and the radiation condition

Solution: (convolution with the Green's function $g = \frac{-1}{2\pi} log|\mathbf{x}|$)

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^2} \frac{-1}{2\pi} log(|\mathbf{x} - \mathbf{y}|)\rho(\mathbf{y})dV_{\mathbf{y}} = g * \rho = V[\rho]$$

Properties of the operator V (2D and 3D):

$$C_0(\mathbb{R}^2) \cap C^{0,\alpha}(\mathbb{R}^2) \to C^{2,\alpha}(\mathbb{R}^2)$$
$$C_0(\mathbb{R}^2) \cap C^{1,\alpha}(\mathbb{R}^2) \to C^{3,\alpha}(\mathbb{R}^2)$$

Electrostatics. Boundary Value Problem

Definition

D is a bounded open region in $\mathbb{R}^3$ with boundary ∂D consisting of a finite number of disjoint, closed bounded surfaces belonging to the class C^2. $\mathbb{R}^3 \backslash \overline{D}$ is assumed to be connected.

Problem

Given $h(\mathbf{x}) \in C(\partial D)$, *find* $\phi(\mathbf{x}) \in C^2(\mathbb{R}^3 \backslash \overline{D}) \cap C(\mathbb{R}^3 \backslash D)$ *that verifies:*

$$\Delta \phi(\mathbf{x}) = 0$$

$$\phi(\mathbf{x})|_{\partial D} = h(\mathbf{x})$$

and the radiation condition: $\phi(\mathbf{x}) = o(1)$ *unif. in all directions* $\frac{\mathbf{x}}{|\mathbf{x}|}$

Uniqueness: (energy estimate: divergence thm. on $F := \phi \nabla \overline{\phi}$)

$$\int_{\mathbb{R}^3 \backslash \overline{D}} |\nabla \phi|^2 dV = \int_{\partial D} \phi \frac{\partial \overline{\phi}}{\partial n} ds$$

Electrostatics. Boundary Value Problem

Existence: (in half plane 2D)

$$\phi(x,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{h(x_0)y}{(x-x_0)^2+y^2}dx_0$$

Identity approximation:

$$\lim_{y\to 0}\frac{1}{\pi}\frac{y}{(x-x_0)^2+y^2} = \delta(x-x_0)$$

...what about the general case?

Electrostatics. Boundary Value Problem

Existence (general case) Notice that:

$$\frac{y}{(x-x_0)^2+y^2} = \frac{\partial}{\partial y} log(\sqrt{(x-x_0)^2+y^2})$$

And also:

$$\frac{1}{\pi}\frac{y}{(x-x_0)^2+y^2} = \frac{-1}{\pi}\frac{\partial}{\partial y_0} log(\sqrt{(x-x_0)^2+(y-y_0)^2})\Big|_{y_0=0}$$

This suggests the use of the following representation for the solution:

$$\phi(\mathbf{x}) = \int_{\partial D} \frac{\partial}{\partial n_{\mathbf{y}}} g(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y})dS_{\mathbf{y}}$$

We get:

$$\phi(\mathbf{x})|_{\partial D} = \frac{\sigma}{2} + D[\sigma] = h(\mathbf{x})$$

where D is a compact operator in some sense.

Electrostatics. Boundary Value Problem

Representation for the solution:

$$\phi(\mathbf{x}) = \int_{\partial D} \frac{\partial}{\partial n_{\mathbf{y}}} g(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y})dS_{\mathbf{y}} - i\int_{\partial D} g(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y})dS_{\mathbf{y}}$$

We get:

$$\phi(\mathbf{x})|_{\partial D} = \frac{\sigma}{2} + D[\sigma] - iS[\sigma] = h(\mathbf{x})$$

where D and S are compact operators in some sense.

$$\frac{\sigma(\mathbf{x})}{2} + \int_{\partial D} \left(\frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|^3} - i\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \right)\sigma(\mathbf{y})dS_{\mathbf{y}} = h(\mathbf{x})$$

Existence of Scattering Problems

General Mechanism:

$$
\begin{array}{ccc}
\text{Uniqueness (PDE+boundary)} & & \text{Existence (PDE+boundary)} \\
\downarrow & & \uparrow \\
\text{Uniqueness Second Kind eq.} & \rightarrow & \text{Existence Second Kind eq.}
\end{array}
$$

Second Kind Integral Equations

Definition

Let X, Y be Banach Spaces. A bounded linear operator $K : X \to Y$ is **compact** iff for each bounded sequence $\{x_n\} \subset X$, the sequence $\{K(x_n)\} \subset Y$ contains a convergent subsequence.

Definition

An operator $L : X \to X$ is second kind if it takes the form $L = I + K$ where I is the identity and K is a compact operator.

Theorem

Let $L : X \to X$ be a second kind operator then exactly one of the following stamens hold:

- *$Lx = 0$ has a non-trivial solution*
- *The operator L has a bounded inverse L^{-1} on X*

Second Kind Integral Equations

Theorem

Let P_n be a sequence of projections on X with $dim(im(P_n)) = n$ such that $||P_n(x) - x|| \to 0 \ \forall x \in X$, and let $Lx = y$ be a second kind integral equation uniquely solvable, then the solutions to

$$P_n L P_n x_n = P_n y$$

verify $||x_n - x|| \to 0$

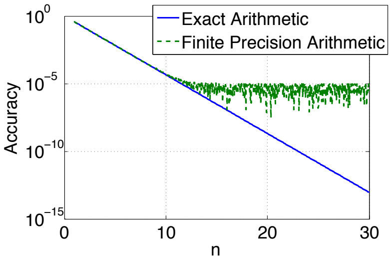
Second Kind Integral Equations

Definition

The condition number of an operator L is: $C := ||L|| \cdot ||L^{-1}||$

Theorem

Solving the equation $P_n L P_n x_n = P_n y$ with finite precision arithmetic, the error saturates to $||x_n - x|| \leq C\epsilon_{machine}$ for $n \geq n_0$



Hölder Spaces

Definition

The Hölder space $C^{0,\alpha}(\partial D)$ is the set of continuous functions $f(\mathbf{x}), \mathbf{x} \in \partial D$ such that $\|f\|_{0,\alpha} < +\infty$ where:

$$\|f\|_{0,\alpha} = \sup_{\mathbf{x}\in\partial D} |f(\mathbf{x})| + \sup_{\mathbf{x},\mathbf{y}\in\partial D,\mathbf{x}\neq\mathbf{y}} \frac{|f(\mathbf{x}) - f(\mathbf{x})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}$$

Theorem

The normed space $\left(C^{0,\alpha}(\partial D), \|\cdot\|_{0,\alpha}\right)$ *is a Banach space.*

Hölder Spaces

Definition

The Hölder space $C^{1,\alpha}(\partial D)$ is the set of differentiable functions $f(x), \mathbf{x} \in \partial D$ such that $\|f\|_{1,\alpha} < +\infty$ where:

$$\|f\|_{1,\alpha} = \|f\|_\infty + \sup_{\mathbf{x} \in \partial D} |\nabla_s f(\mathbf{x})| + \sup_{\mathbf{x},\mathbf{y} \in \partial D, \mathbf{x} \neq \mathbf{y}} \frac{|\nabla_s f(\mathbf{x}) - \nabla_s f(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\alpha}$$

Theorem

The normed space $\left(C^{1,\alpha}(\partial D), \|\cdot\|_{1,\alpha}\right)$ is a Banach space.

Hölder Spaces

Theorem

For any $0 < \alpha < \beta \leq 1$ and $0 < \gamma < \delta \leq 1$ The following imbeddings are compact:

$$C^{1,\beta}(\partial D) \to C^{1,\alpha}(\partial D) \to C^{0,\delta}(\partial D) \to C^{0,\gamma}(\partial D) \to C(\partial D)$$

e.g:

$$T_1 : C(\partial D) \to_{cont} C^{0,\gamma}(\partial D) \qquad \Rightarrow T_1 : C(\partial D) \to_{comp} C(\partial D)$$

$$\begin{aligned} T_1 &: C^{1,\alpha}(\partial D) \to_{cont} C^{0,\delta}(\partial D) \\ T_2 &: C(\partial D) \to_{cont} C^{1,\alpha}(\partial D) \qquad \Rightarrow T_1 T_2 : C(\partial D) \to_{comp} C(\partial D) \end{aligned}$$

Layer Potentials:

$$S[\sigma](\mathbf{x}) = \int_{\partial D} g(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y})dA_y, \;\; D[\sigma](\mathbf{x}) = \int_{\partial D} \frac{\partial g}{\partial n_y}(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y})dA_y$$

Theorem

Let $g(\mathbf{x},\mathbf{y})$ defined and continuous for all $\mathbf{x},\mathbf{y} \in \partial D, \mathbf{x} \neq \mathbf{y}$, and there are constants $M > 0, \alpha \in (0,2]$ such that:

$$|g(\mathbf{x},\mathbf{y})| \leq M|\mathbf{x}-\mathbf{y}|^{\alpha-2}, \quad \mathbf{x} \neq \mathbf{y}$$

Then the operator $h(\mathbf{x}) = \int_{\partial D} g(\mathbf{x},\mathbf{y}) f(\mathbf{y}) dA_{\mathbf{y}}$ is compact on $C(\partial D)$

Layer Potentials

Surface operator ($\mathbf{x} \in \partial D$)	Regularity (continuous map)
$S\sigma = \int_{\partial D} g(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y})dA_y$	$\left\{ \begin{array}{c} C(\partial D) \to C^{0,\alpha}(\partial D) \\ C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D) \end{array} \right.$
$D\sigma = \int_{\partial D} \frac{\partial g}{\partial n_y}(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y})dS_y$	$\left\{ \begin{array}{c} C(\partial D) \to C^{0,\alpha}(\partial D) \\ C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D) \end{array} \right.$
$S'\sigma = \int_{\partial D} \frac{\partial g}{\partial n_x}(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y})dS_y$	$\left\{ \begin{array}{c} C(\partial D) \to C^{0,\alpha}(\partial D) \\ C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D) \end{array} \right.$
$D'\sigma = \frac{\partial}{\partial n_x} \int_{\partial D} \frac{\partial g}{\partial n_y}(\mathbf{x}-\mathbf{y})\sigma(\mathbf{y})dS_y$	$C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$

Existence of Scattering Problems

Problem

Given $h(\mathbf{x}) \in C(\partial D)$*, find* $\phi(\mathbf{x}) \in C^2(\mathbb{R}^3 \backslash \overline{D}) \cap C(\mathbb{R}^3 \backslash D)$ *that verifies:*

$$\Delta \phi(\mathbf{x}) = 0$$

$$\phi(\mathbf{x})|_{\partial D} = h(\mathbf{x})$$

and the radiation condition: $\phi(\mathbf{x}) = o(1)$ *unif. in all directions* $\frac{\mathbf{x}}{|\mathbf{x}|}$

Theorem

$$\phi^*(\mathbf{x}) = \int_{\partial D} \left(\frac{\partial g}{\partial n_y}(\mathbf{x}-\mathbf{y}) - ig(\mathbf{x}-\mathbf{y}) \right) \sigma(\mathbf{y}) dS_y =$$

The function ϕ^* *verifies the exterior Dirichlet problem provided* σ *verifies the following second kind integral equation*

$$\frac{\sigma}{2} + D\sigma - iS\sigma = h$$

Existence of Scattering Problems

Theorem

The exterior Dirichlet problem has unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on $\mathbb{R}^3 \backslash D$ and all its derivatives on closed subsets of $\mathbb{R}^3 \backslash \overline{D}$

$$||\phi^{\text{scat}}||_{\infty,\mathbb{R}^3\backslash\overline{D}} \leq C_{(\partial D)}||h||_{\infty,\partial D}$$

Integral Equation

$$\frac{\sigma(\mathbf{x})}{2} + \int_{\partial D} \left(\frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{4\pi|\mathbf{x} - \mathbf{y}|^3} - i\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \right) \sigma(\mathbf{y}) dS_{\mathbf{y}} = h(\mathbf{x})$$

Numerical solution using the Nyström method.

Definition

Given $h(\mathbf{x}) \in C(\partial D)$, find $\phi(\mathbf{x}) \in C^2(\mathbb{R}^3 \backslash \overline{D}) \cap C(\mathbb{R}^3 \backslash D)$ that verifies:

$$\Delta\phi(\mathbf{x}) + k^2\phi(\mathbf{x}) = 0$$
$$\phi(\mathbf{x})|_{\partial D} = h(\mathbf{x})$$

$(k > 0)$ and verifies the radiation condition:

$$\frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla\phi(\mathbf{x}) + -ik\phi(\mathbf{x}) = o\left(\frac{1}{|\mathbf{x}|}\right), \quad |\mathbf{x}| \to \infty$$

Uniformly in all directions $\frac{\mathbf{x}}{|\mathbf{x}|}$.

Theorem

Let $\phi^{scat}(\mathbf{x}) \in C^2(\mathbb{R}^3 \backslash \overline{D}) \cap C(\mathbb{R}^3 \backslash D)$ *be a Helmholtz scalar function* $\Delta\phi + k^2\phi = 0$ *and satisfying the R.C. and:*

$$I_s = \Im\left\{ k \int_{\partial D} \phi \frac{\partial \overline{\phi}}{\partial n} ds \right\} \geq 0$$

then, $\phi = 0$ *in* $\mathbb{R}^3 \backslash \overline{D}$. $(k > 0)$

Similar approach using the Green's function $g_k(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}$

$$\phi(\mathbf{x}) = \int_{\partial D} \left(\frac{\partial g_k}{\partial n_y}(\mathbf{x} - \mathbf{y}) - i g_k(\mathbf{x} - \mathbf{y}) \right) \sigma(\mathbf{y}) dS_y =$$

ϕ verifies the partial differential equation, the radiation condition by construction.

Existence of Scattering Problems

Theorem

The exterior Dirichlet problem has unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on $\mathbb{R}^3 \backslash D$ and all its derivatives on closed subsets of $\mathbb{R}^3 \backslash \overline{D}$

$$||\phi^{\text{scat}}||_{\infty,\mathbb{R}^3\backslash\overline{D}} \leq C_{(\partial D,k)}||f||_{\infty,\partial D}$$

Theorem

The continuity is uniform on any $k \in [0, k_{\text{max}}]$

$$||\phi^{\text{scat}}||_{\infty,\mathbb{R}^3\backslash\overline{D}} \leq C_{(\partial D,k_{\text{max}})}||f||_{\infty,\partial D}$$

Theorem

The operators S_k, D_k are collectively compact on $C(\partial D)$ for $k \in [0, k_{\text{max}}]$

Integral Equation

$$\frac{\sigma(\mathbf{x})}{2} + \int_{\partial D} \left(\frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{4\pi|\mathbf{x} - \mathbf{y}|^3} - i\eta \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \right) e^{ik|\mathbf{x}-\mathbf{y}|} \sigma(\mathbf{y}) dS_{\mathbf{y}} = f(\mathbf{x})$$

Using the Nyström method (locally corrected for the diagonal terms) we get uniform accuracy for $k \in [0, k_{\max}]$.

Definition

Let $\mathbf{h} \in T_d^{0,\alpha}(\partial D)$.
Find $\mathbf{E}^{\mathbf{scat}}(\mathbf{x})$, $\mathbf{H}^{\mathbf{scat}}(\mathbf{x}) \in C^1(\mathbb{R}^3 \backslash \overline{D}) \cap C(\mathbb{R}^3 \backslash D)$ such that:

$$\begin{aligned} \nabla \times \mathbf{E}^{\mathbf{scat}} &= ik\mathbf{H}^{\mathbf{scat}}, \qquad \mathbf{n} \times \mathbf{E}^{\mathbf{scat}}|_{\partial D} = \mathbf{h} \\ \nabla \times \mathbf{H}^{\mathbf{scat}} &= -ik\mathbf{E}^{\mathbf{scat}}, \end{aligned}$$

$(k > 0)$ and verifies the radiation condition:

$$\mathbf{H}^{\mathbf{scat}}(\mathbf{x}) \times \frac{\mathbf{x}}{|\mathbf{x}|} - \mathbf{E}^{\mathbf{scat}}(\mathbf{x}) = o\left(\frac{1}{|\mathbf{x}|}\right), \quad |\mathbf{x}| \to \infty$$

Uniformly in all directions $\frac{\mathbf{x}}{|\mathbf{x}|}$.

Vector Hölder Spaces

Definition

The Hölder space $T^{0,\alpha}(\partial D)$ is the set of tangent fields $f(\mathbf{x}), \mathbf{x} \in \partial D$ such that $\|\mathbf{f}\|_{T^{0,\alpha}} < +\infty$ where:

$$\|\mathbf{f}\|_{T^{0,\alpha}} = \sup_{\mathbf{x}\in\partial D} |\mathbf{f}(\mathbf{x})| + \sup_{\mathbf{x},\mathbf{y}\in\partial D,\mathbf{x}\neq\mathbf{y}} \frac{|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}$$

Theorem

The normed space $\left(T^{0,\alpha}(\partial D), \|\cdot\|_{T^{0,\alpha}}\right)$ *is a Banach space.*

Vector Hölder Spaces

Definition

The Hölder space $T_d^{0,\alpha}(\partial D)$ is the set of tangent fields possessing surface divergence such that $\|\mathbf{f}\|_{T_d^{0,\alpha}} < +\infty$ where:

$$\|\mathbf{f}\|_{T_d^{0,\alpha}} = \|\mathbf{f}(\mathbf{x})\|_{T^{0,\alpha}} + \|\nabla_s \cdot \mathbf{f}\|_{0,\alpha}$$

Theorem

The normed space $\left(T_d^{0,\alpha}(\partial D), \|\cdot\|_{T_d^{0,\alpha}}\right)$ *is a Banach space.*

Vector Hölder Spaces

Definition

The Hölder space $T_r^{0,\alpha}(\partial D)$ is the set of tangent fields possessing surface divergence such that $\|\mathbf{f}\|_{T_r^{0,\alpha}} < +\infty$ where:

$$\|\mathbf{f}\|_{T_r^{0,\alpha}} = \|\mathbf{f}(\mathbf{x})\|_{T^{0,\alpha}} + \|\nabla_s \cdot (\mathbf{n} \times \mathbf{f})\|_{0,\alpha}$$

Theorem

The normed space $\left(T_r^{0,\alpha}(\partial D), \|\cdot\|_{T_r^{0,\alpha}}\right)$ *is a Banach space.*

Vector Hölder Spaces

Theorem

For any $0 < \alpha < \beta \leq 1$ *and* $0 < \gamma < \delta \leq 1$ *The following imbeddings are compact:*

$$T_r^{0,\beta}(\partial D) \to T_r^{0,\alpha}(\partial D) \to T^{0,\delta}(\partial D) \to T^{0,\gamma}(\partial D) \to T(\partial D)$$

$$T_d^{0,\beta}(\partial D) \to T_d^{0,\alpha}(\partial D) \to T^{0,\delta}(\partial D) \to T^{0,\gamma}(\partial D) \to T(\partial D)$$

Layer Potentials

Green's Function in the free space:

$$g_k(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}$$

Layer Potentials:

$$S_k[\mathbf{J}](\mathbf{x}) = \int_{\partial D} g_k(\mathbf{x}-\mathbf{y})\mathbf{J}(\mathbf{y})dA_y$$

$$\nabla \times S_k[\mathbf{J}](\mathbf{x}) = \nabla \times \int_{\partial D} g_k(\mathbf{x}-\mathbf{y})\mathbf{J}(\mathbf{y})dA_y$$

$$\nabla \times \nabla \times S_k[\mathbf{J}](\mathbf{x}) = \nabla \times \nabla \times \int_{\partial D} g_k(\mathbf{x}-\mathbf{y})\mathbf{J}(\mathbf{y})dA_y$$

Layer Potentials

Surface operator $(\mathbf{x} \in \partial D)$	Regularity (continuous map)
$\mathbf{n} \times S_k(\mathbf{J})$	$\begin{cases} T(\partial D) \to T^{0,\alpha}(\partial D) \\ T^{0,\alpha}(\partial D) \to T_d^{0,\alpha}(\partial D) \end{cases}$
$M_k(\mathbf{J}) = \int_{\partial D} \mathbf{n}_x \times \nabla_x \times (g_k(\mathbf{x}-\mathbf{y})\mathbf{J}(\mathbf{y}))dA_y$	$T(\partial D) \to T^{0,\alpha}(\partial D)$
$T_k(\mathbf{J})$	$T_d^{0,\alpha}(\partial D) \to T_d^{0,\alpha}(\partial D)$

where: $T_k(\mathbf{J}) = \int_{\partial D} \mathbf{n}_x \times \nabla_x \times \nabla_x \times (g_k(\mathbf{x}-\mathbf{y})\mathbf{J}(\mathbf{y}))dA_y$

Theorem

Let $\mathbf{E}^{\text{scat}}(\mathbf{x})$, $\mathbf{H}^{\text{scat}}(\mathbf{x})$ *be a Maxwellian field* $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$ *(R.C.):*

$$I_s = \Re\left\{ \int_{\partial D} \mathbf{n} \times \mathbf{E}^{\text{scat}} \cdot \overline{\mathbf{H}}^{\text{scat}} ds \right\} \leq 0$$

then, $\mathbf{E}^{\text{scat}}(\mathbf{x}) = \mathbf{H}^{\text{scat}}(\mathbf{x}) = 0$

Guess:

$$\mathbf{E}(\mathbf{x}) = \nabla \times S_k[\mathbf{J}](\mathbf{x}) + i\nabla \times \nabla \times S_k[\mathbf{n} \times S_0^2(\mathbf{J})](\mathbf{x})$$

$$\mathbf{H}(\mathbf{x}) = \frac{1}{ik}\nabla \times \mathbf{E}(\mathbf{x})$$

$\mathbf{E}, \mathbf{H}$ verifies the Maxwell's equations and the radiation condition by construction.
Boundary condition:

$$\mathbf{E}(\mathbf{x}) = \nabla \times S_k[\mathbf{J}](\mathbf{x}) + i\nabla \times \nabla \times S_k[\mathbf{n} \times S_0^2(\mathbf{J})](\mathbf{x})$$

$$\mathbf{n} \times \mathbf{E}(\mathbf{x})|_{\partial D} = \mathbf{f} = \frac{\mathbf{J}}{2} + \mathbf{M_k}(\mathbf{J}) + i\mathbf{T_k}(\mathbf{n} \times \mathbf{S_0^2}(\mathbf{J}))$$

Low frequency stability

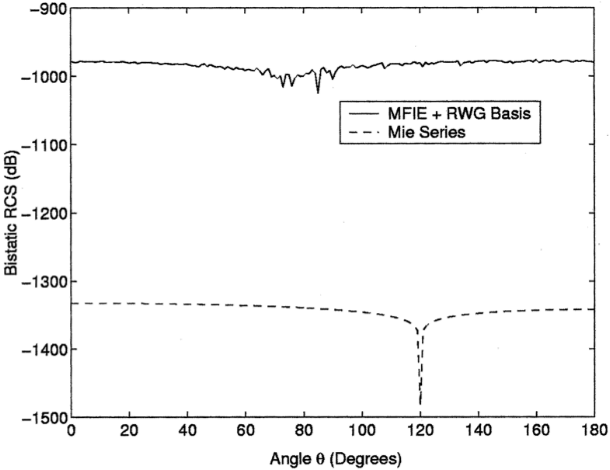
Condition number of the resulting integral equations:

$$cond\left(\frac{I}{2} + D_k - iS_k\right) \leq C_{k_{max}}, \forall k \in]0, k_{max}]$$

$$cond\left(\frac{\mathbf{I}}{2} + \mathbf{M_k} + i\eta \mathbf{T_k} \mathbf{n} \times \mathbf{S_0^2}\right) \xrightarrow{k \to 0} +\infty$$

- This problem is called **low frequency breakdown** in the literature
- Almost all integral equations proposed for Electromagnetics have the same problem, in one way or another

Magnetic Field Integral Equation (MFIE):
[Chew (2003) IEEE TAP]



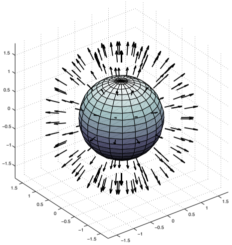
Nullspace of Maxwell's equations in static regime ($k = 0$)

Definition

Find $\mathbf{E}^{\text{scat}}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$ such that (and RC):

$$\begin{aligned} \nabla \times \mathbf{E}^{\text{scat}} &= \mathbf{0}, \\ \nabla \cdot \mathbf{E}^{\text{scat}} &= 0, \\ \mathbf{n} \times \mathbf{E}^{\text{scat}}|_{\partial D} &= \mathbf{h} \end{aligned}$$

Dirichlet fields (nullspace):

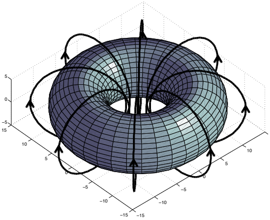


Definition

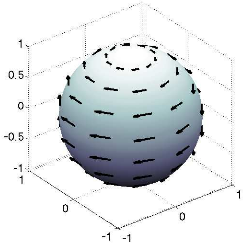
Find $\mathbf{H}^{\text{scat}}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$ such that (and RC):

$$\begin{aligned} \nabla \times \mathbf{H}^{\text{scat}} &= \mathbf{0}, \\ \nabla \cdot \mathbf{H}^{\text{scat}} &= 0, \\ \mathbf{n} \cdot \mathbf{H}^{\text{scat}}|_{\partial D} &= -\mathbf{n} \cdot \mathbf{H}^{\text{inc}}|_{\partial D} \end{aligned}$$

Neumann fields (nullspace):



Non-Existence of solution in static regime ($k = 0$)

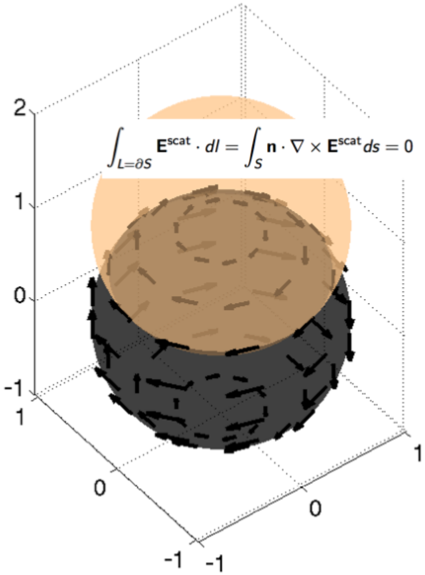


$$\mathbf{E}_0^{\text{scat}} = -\nabla \phi_0^{\text{scat}}$$

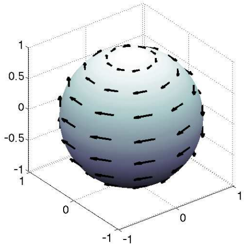
$$\mathbf{n} \times \nabla \phi_0^{\text{scat}} = -\mathbf{n} \times \nabla \phi_0^{\text{inc}}$$

$$\Delta \phi_0^{\text{scat}} = 0$$

$$\phi_0^{\text{scat}} = -\phi_0^{\text{inc}}$$



Non-Existence of solution in static regime ($k = 0$)



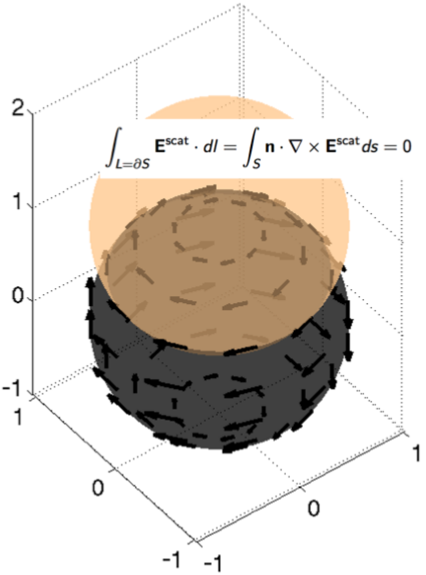
$$\mathbf{E}_0^{\text{scat}} = -\nabla\phi_0^{\text{scat}}$$

$$\mathbf{n}\times\nabla\phi_0^{\text{scat}} = -\mathbf{n}\times\nabla\phi_0^{\text{inc}}$$

$$\Delta\phi_0^{\text{scat}} = 0$$

$$\phi_0^{\text{scat}} = -\phi_0^{\text{inc}} + V_j$$

$$\int_{\partial D_j}\frac{\partial\phi_0^{\text{scat}}}{\partial n}\,ds = Q_j$$



The Electric Extended Problem

Definition

Let $\mathbf{h} \in T_d^{0,\alpha}(\partial D)$,$f \in C_d^{0,\alpha}(\partial D)$, and q_j constants
Find $\mathbf{E}(\mathbf{x}) \in C^2(\mathbb{R}^3 \backslash \overline{D}) \cap C(\mathbb{R}^3 \backslash D)$ such that:
$\nabla \cdot \mathbf{E}, \nabla \times \mathbf{E}(\mathbf{x}) \in C(\mathbb{R}^3 \backslash D)$ and v_j

$$\Delta \mathbf{E} + k^2 \mathbf{E} = 0$$

$$\mathbf{n} \times \mathbf{E}|_{\partial D} = \mathbf{h}|_{\partial D}$$

$$\nabla \cdot \mathbf{E}|_{\partial D_j} = f + v_j$$

$$\int_{\partial D_j} \mathbf{n} \cdot \mathbf{E}^{\text{scat}} ds = q_j$$

$$\nabla \times \mathbf{E} \times \frac{x}{|x|} + \frac{x}{|x|} \nabla \cdot \mathbf{E} - ik\mathbf{E} = o\left(\frac{1}{|\mathbf{x}|}\right), \quad |\mathbf{x}| \to \infty$$

Uniformly in all directions $\frac{\mathbf{x}}{|\mathbf{x}|}$.

The Electric Extended Problem

If **E** verifies:

$$\begin{cases} \Delta \mathbf{E} + k^2 \mathbf{E} = 0 \\ \mathbf{n} \times \mathbf{E}|_{\partial D} = \mathbf{h} \\ \nabla \cdot \mathbf{E}|_{\partial D_j} = 0 + v_j \\ \int_{\partial D_j} \mathbf{n} \cdot \mathbf{E}^{\text{scat}} ds = 0 \end{cases}$$

Then $\phi := \nabla \cdot \mathbf{E}$ verifies:

$$\begin{cases} \Delta \phi + k^2 \phi = 0 \\ \phi|_{\partial D_j} = 0 + V_j \\ \int_{\partial D_j} \frac{\partial \phi}{\partial n} ds = 0 \end{cases}$$

Then $\nabla \cdot \mathbf{E} = 0$ verifies:

The Electric Extended Problem

Energy Estimate for vector Helmholtz fields

Theorem

Let $\mathbf{E}(\mathbf{x})$ *be a Helmholtz field* $\Delta \mathbf{E} + k^2 \mathbf{E} = 0$ $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$ *(RC):*

$$I_s = \Im \left\{ k \int_{\partial D} \mathbf{n} \times \mathbf{E} \cdot \nabla \times \overline{\mathbf{E}} + \mathbf{n} \cdot \mathbf{E} \nabla \cdot \overline{\mathbf{E}} ds \right\} \geq 0$$

then, $\mathbf{E} = 0$ $(k > 0)$

The Electric Extended Problem

Representation for the Vector potential:

$$\mathbf{E}(\mathbf{x}) = \nabla \times S_k[\mathbf{a}] - S_k(\mathbf{n}\rho) + i\eta\big(S_k[\mathbf{n} \times S_{ik}[\mathbf{a}]](\mathbf{x}) + \nabla S_k[S_{ik}[\rho]](\mathbf{x})\big)$$

Second kind integral equation:

$$\frac{1}{2}\begin{bmatrix} \mathbf{a} \\ \rho \end{bmatrix} + \overline{\overline{L}}\begin{bmatrix} \mathbf{a} \\ \rho \end{bmatrix} + i\eta\overline{\overline{R}} \cdot \overline{\overline{S_{ik}}}\begin{bmatrix} \mathbf{a} \\ \rho \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{j=1}^{N} v_j \chi_j \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ h \end{bmatrix},$$

$$\int_{\partial D_j} \left(\mathbf{n} \cdot \nabla \times S_k \mathbf{a} - \mathbf{n} \cdot S_k(\mathbf{n}\rho) + i\eta\big(\mathbf{n} \cdot S_k(\mathbf{n} \times S_{ik}\mathbf{a}) - \tfrac{\rho}{2} + S_k'\rho\big)\right) ds = q_j,$$

Stability of the Extended Electric Problem

Theorem

The Electric Extended problem has unique solution for $k \geq 0$ and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on $\mathbb{R}^3 \backslash D$ and all its derivatives on closed subsets of $\mathbb{R}^3 \backslash \overline{D}$

$$||\mathbf{E}||_{\infty,\mathbb{R}^3\backslash\overline{D}} \leq C_{(\partial D,k)} \left(||\mathbf{h}||_{\infty,\partial D} + ||f||_{\infty,\partial D} + \sqrt{\sum_{j=1}^{N} |q_j|^2} \right)$$

Theorem

The continuity is uniform on any $k \in [0, k_{\max}]$

$$||\mathbf{E}||_{\infty,\mathbb{R}^3\backslash\overline{D}} \leq C_{(\partial D,k_{max})} \left(||\mathbf{h}||_{\infty,\partial D} + ||f||_{\infty,\partial D} + \sqrt{\sum_{j=1}^{N} |q_j|^2} \right)$$

Compatibility with Maxwell's Equations

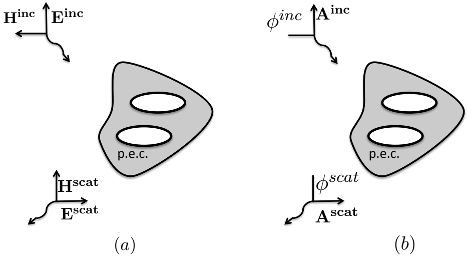
Theorem

If $\mathbf{h} = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}}$ *where* $\mathbf{E}^{\mathrm{inc}}$ *is a Maxwellian vector field and* $f = 0$ *and* $q_j = 0$*, then, the solution of the Extended Electric Problem is a Maxwellian field*

Decoupled Potentials boundary conditions

Suitable boundary conditions (for any $\omega \geq 0$) on the potentials separately would solve the bad scaling problem and the non uniqueness of the static limit ($d\mathcal{F} = 0 \Rightarrow \mathcal{F} = d\mathcal{A}$):

$$\begin{aligned} \mathbf{n} \times \mathbf{A}^{\text{scat}}(\mathbf{x}) &= -\mathbf{n} \times \mathbf{A}^{\text{inc}}(\mathbf{x})|_{\partial D} \\ \mathbf{n} \times \nabla\phi^{\text{scat}}(\mathbf{x}) &= -\mathbf{n} \times \nabla\phi^{\text{inc}}(\mathbf{x})|_{\partial D} \end{aligned}$$



[4] Vico, F., Ferrando, M., Greengard, L., Gimbutas, Z. (2016). The Decoupled Potential Integral Equation for Time-Harmonic Electromagnetic Scattering. Communications on Pure and Applied Mathematics, 69(4), 771-812.

Boundary conditions

Table: Decoupled boundary value problems for the vector and scalar potentials. Uniquely solvable for $\omega \geq 0$. Unknowns: ϕ^{scat}, $\{V_j\}_{j=1}^N$ and $\mathbf{A}^{\text{scat}}$, $\{v_j\}_{j=1}^N$. The surface boundary is: $\partial D = \partial D_1 \cup \partial D_2 \cup ... \cup \partial D_n$

	Modified Dirichlet Problem
Scalar	$\begin{cases} \Delta \phi^{\text{scat}} + k^2 \phi^{\text{scat}} = 0 \\ \phi^{\text{scat}}\|_{\partial D_j} = -\phi^{\text{inc}}\|_{\partial D_j} + V_j \\ \int_{\partial D_j} \frac{\partial \phi^{\text{scat}}}{\partial n} ds = -\int_{\partial D_j} \frac{\partial \phi^{\text{inc}}}{\partial n} ds \end{cases}$
Vector	$\begin{cases} \Delta \mathbf{A}^{\text{scat}} + k^2 \mathbf{A}^{\text{scat}} = 0 \\ \mathbf{n} \times \mathbf{A}^{\text{scat}}\|_{\partial D} = -\mathbf{n} \times \mathbf{A}^{\text{inc}}\|_{\partial D} \\ \nabla \cdot \mathbf{A}^{\text{scat}}\|_{\partial D_j} = -\nabla \cdot \mathbf{A}^{\text{inc}}\|_{\partial D_j} + v_j \\ \int_{\partial D_j} \mathbf{n} \cdot \mathbf{A}^{\text{scat}} ds = -\int_{\partial D_j} \mathbf{n} \cdot \mathbf{A}^{\text{inc}} ds \end{cases}$

Theorems about Modified Dirichlet problems:

Theorem

For any $k \geq 0$ the Scalar Modified Dirichlet Problem has one and only one solution.

Theorem

For any $k \geq 0$ the Vector Modified Dirichlet Problem has one and only one solution.

Fundamental theorem:

Theorem

For any $\omega \geq 0$, let $\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}}$ be an incoming electromagnetic field described by the 4-potential $\mathbf{A}^{\text{inc}}, \phi^{\text{inc}}$ in the Lorenz gauge. That is:

$$\begin{aligned} \mathbf{H}^{\text{inc}} &= \mu^{-1}\nabla \times \mathbf{A}^{\text{inc}} \\ \mathbf{E}^{\text{inc}} &= i\omega\mathbf{A}^{\text{inc}} - \nabla\phi^{\text{inc}} \\ \nabla \cdot \mathbf{A}^{\text{inc}} &= i\omega\mu\epsilon\phi^{\text{inc}} . \end{aligned}$$

Then $\mathbf{E}^{\text{scat}}, \mathbf{H}^{\text{scat}}$, can be described by $\mathbf{A}^{\text{scat}}, \phi^{\text{scat}}$. More precisely,

$$\begin{aligned} \mathbf{H}^{\text{scat}} &= \mu^{-1}\nabla \times \mathbf{A}^{\text{scat}} \\ \mathbf{E}^{\text{scat}} &= i\omega\mathbf{A}^{\text{scat}} - \nabla\phi^{\text{scat}} \\ \nabla \cdot \mathbf{A}^{\text{scat}} &= i\omega\mu\epsilon\phi^{\text{scat}} \end{aligned}$$

Scheme of the solution:

$$\begin{tikzcd}
& (E^{inc}, H^{inc}) \arrow[r, dashed] & (E^{scat}, H^{scat}) \\
(J^{imp}, \rho^{imp}) \arrow[ur] \arrow[dr] & & \\
& (A^{inc}, \phi^{inc}) \arrow[uu] \arrow[r] & (A^{scat}, \phi^{scat}) \arrow[uu]
\end{tikzcd}$$

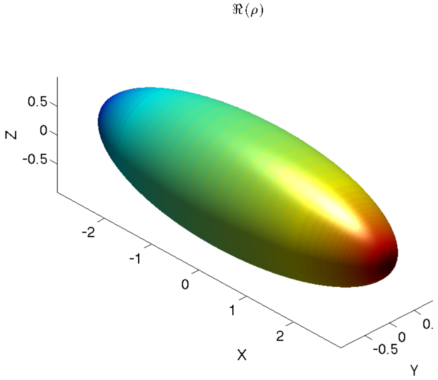
Theorem

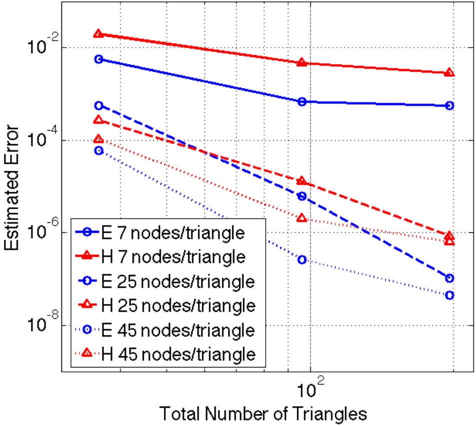
The four potential $\mathbf{A}^{\text{scat}}, \phi^{\text{scat}}$*, solution to the modified Dirichlet problem depends continuously on the boundary data* $\mathbf{A}^{\text{inc}}, \phi^{\text{inc}}$ *on the surface. This continuity is uniform for* $\omega \in [0, \omega_{max}]$

$$\begin{array}{rcl} f, \{Q_j\}_{j=1}^N \in C(\partial D) \times \mathbb{C}^N & \to & \phi^{\text{scat}}, \{V_j\}_{j=1}^N \in C(\mathbb{R}^3/D) \times \mathbb{C}^N \\ \mathbf{f}, h, \{Q_j\}_{j=1}^N \in T(\partial D) \times C(\partial D) \times \mathbb{C}^N & \to & \mathbf{A}^{\text{scat}}, \{v_j\}_{j=1}^N \in C(\mathbb{R}^3/D) \times \mathbb{C}^N \end{array}$$

for any $k \in [0, k_{max}]$*, with fixed* k_{max}.

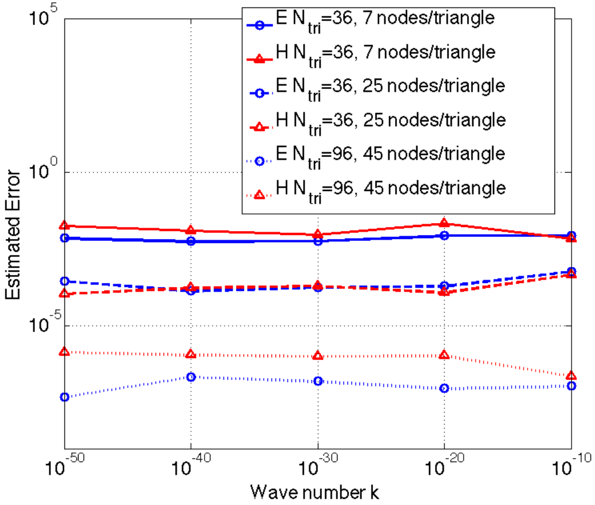
DPIE. Frontal solver. High Order Nyström. Convergence analysis.
$\mathbf{A}^{\mathbf{inc}} = -xe^{ikz}\mathbf{z}$ $\phi^{\mathbf{inc}} = -xe^{ikz}$, $(\mathbf{E}^{\mathbf{inc}} = e^{ikz}\mathbf{x}, \mathbf{H}^{\mathbf{inc}} = e^{ikz}\mathbf{y})$, $k = 0.1$



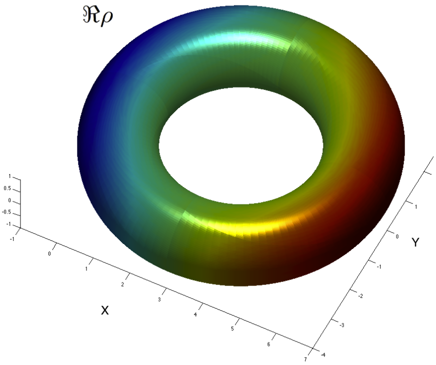


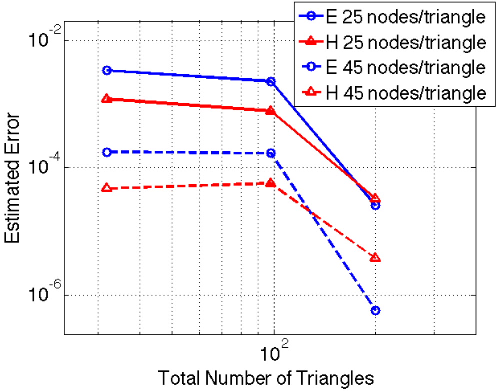
Numerical Results. Ellipsoid 3x1x1

DPIE . Frontal solver. High Order Nyström. Low frequency stability



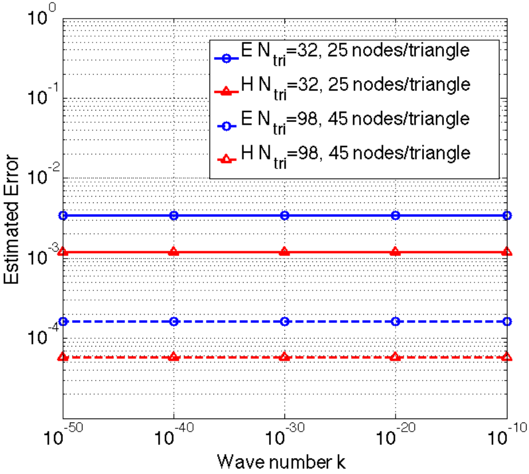
DPIE . Frontal solver. High Order Nyström. Convergence analysis.
$\mathbf{A}^{\mathbf{inc}} = -xe^{ikz}\mathbf{z}$ $\phi^{\mathbf{inc}} = -xe^{ikz}$, $(\mathbf{E}^{\mathbf{inc}} = e^{ikz}\mathbf{x}, \mathbf{H}^{\mathbf{inc}} = e^{ikz}\mathbf{y})$
$k = 10^{-3}$





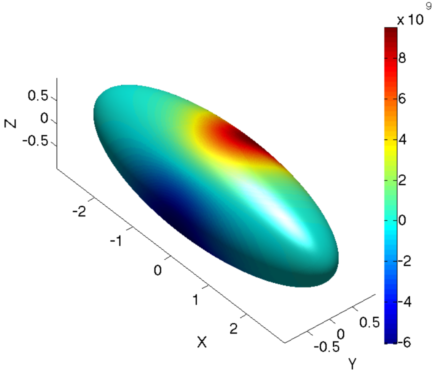
Numerical Results. Torus

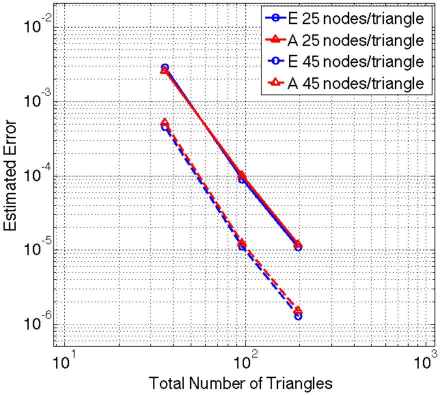
DPIE . Frontal solver. High Order Nyström. Low frequency stability



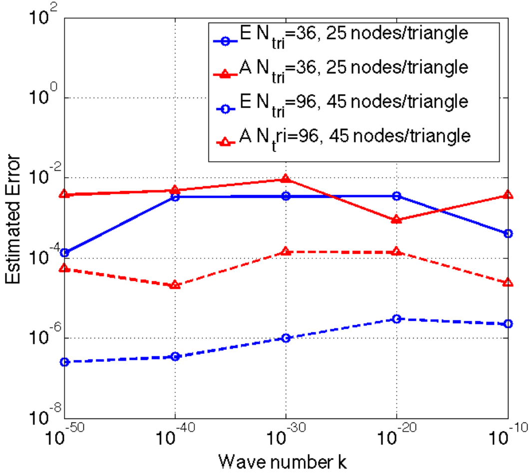
Numerical Results. Ellipsoid 3x1x1

DPIE . Frontal solver. High Order Nyström. Convergence analysis. Source located inside (small loop and dipole linear comb.). $k = 0.1$



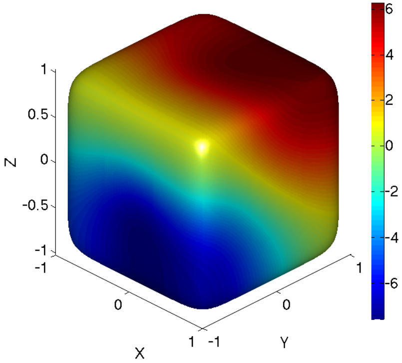


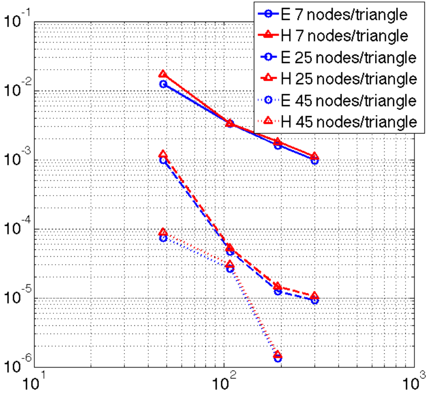
DPIE (physical **non – physical**). Frontal solver. High Order Nyström. Low frequency stability



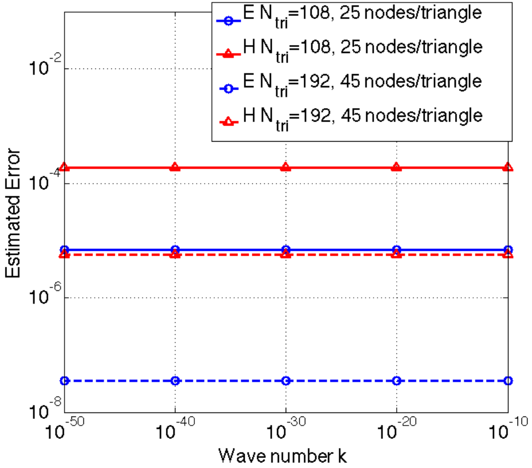
Numerical Results. Superellipsoid $x^8 + y^8 + z^8 = 1$

DPIE . Frontal solver. High Order Nyström. Convergence analysis. Source located inside (small loop and dipole linear comb.). $k = 0.1$



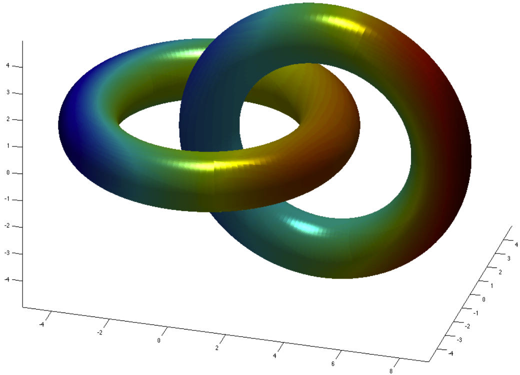


DPIE (physical **non – physical**). Frontal solver. High Order Nyström. Low frequency stability



Double Torus

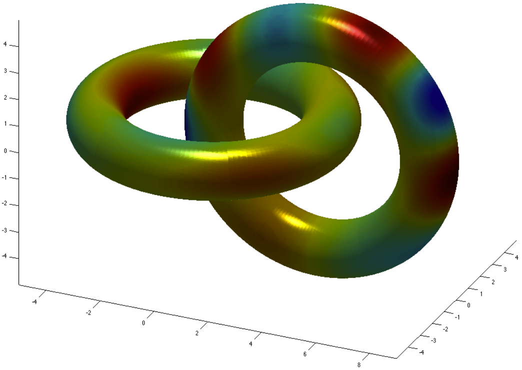
Decoupled Potential Integral Equation. Frontal solver. High Order Nyström



- $N_{triangles} = 100$
- $N_{unknownsDPIEv} = 13502$
- $N_{unknownsDPIEs} = 4502$
- Nodes per triangle $= 45$
- $D_{max} = 1.6 \cdot 10^{-13}\lambda$
- $\mathbf{A}^{\mathbf{inc}} = -xe^{ikz}\mathbf{z}, \phi^{\mathbf{inc}} = -xe^{ikz}$,
- Condition Num.$_{DPIEv} = 49$
- Error $\mathbf{E}^{\mathbf{scat}} = 3.5 \cdot 10^{-3}$
- Error $\mathbf{H}^{\mathbf{scat}} = 2.4 \cdot 10^{-3}$

Double Torus

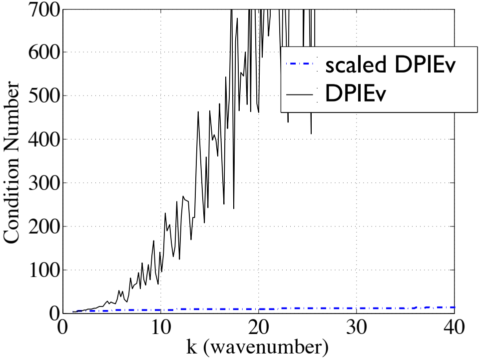
Decoupled Potential Integral Equation. Frontal solver. High Order Nyström



- $N_{triangles} = 144$
- $N_{unknownsDPIEv} = 19442$
- $N_{unknownsDPIEs} = 6482$
- Nodes per triangle $= 45$
- $D_{max} = 2.39\lambda$
- $\mathbf{A}^{\mathbf{inc}} = -xe^{ikz}\mathbf{z}, \phi^{\mathbf{inc}} = -xe^{ikz}$,
- Condition Num.$_{DPIEv}$ = n.a.
- Error $\mathbf{E}^{\mathbf{scat}} = 5.2 \cdot 10^{-4}$
- Error $\mathbf{H}^{\mathbf{scat}} = 2.7 \cdot 10^{-4}$

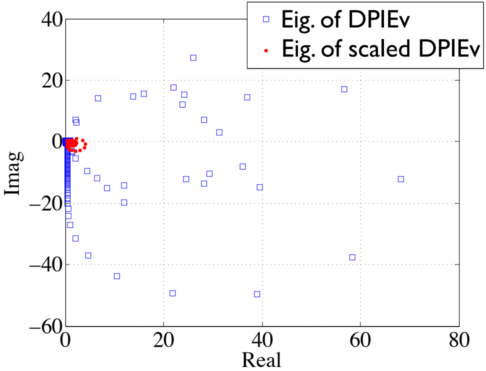
High Frequency Regime

Decoupled potential integral equation with and without rescaling:

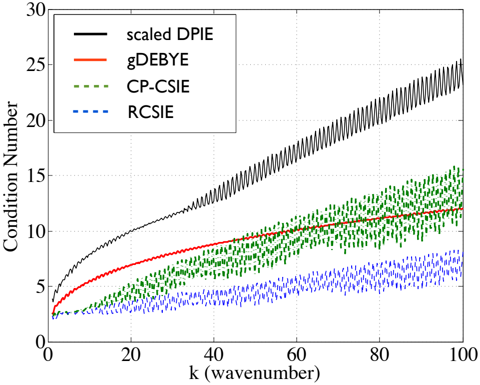


High Frequency Regime

Eigenvalues of the Decoupled potential integral equation with and without rescaling:



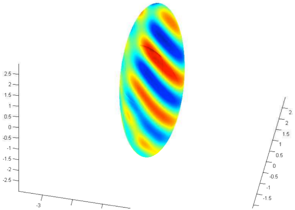
Eigenvalues of the Decoupled potential integral equation with and without rescaling:

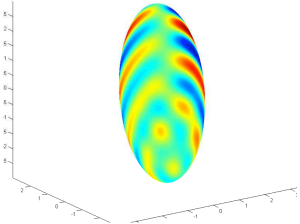


Numerical Experiments (High Frequency)

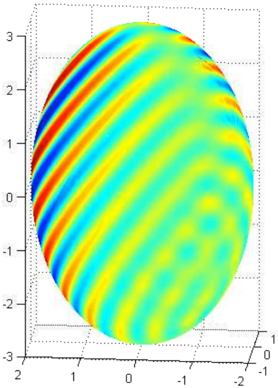
$N_{triangles}$	36	76	116
$N_{unknowns}$	4860	10260	15660
Time	108 s	222 s	335 s
Error $\mathbf{E}^{\mathbf{scat}}$	$9.2 \cdot 10^{-3}$	$1.5 \cdot 10^{-4}$	$7.9 \cdot 10^{-6}$
Error $\mathbf{H}^{\mathbf{scat}}$	$7.7 \cdot 10^{-3}$	$7.4 \cdot 10^{-5}$	$7.1 \cdot 10^{-6}$
k	5	5	5

8th order polynomials
45 points/triangle



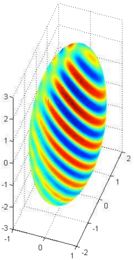


Numerical Experiments (High Frequency)

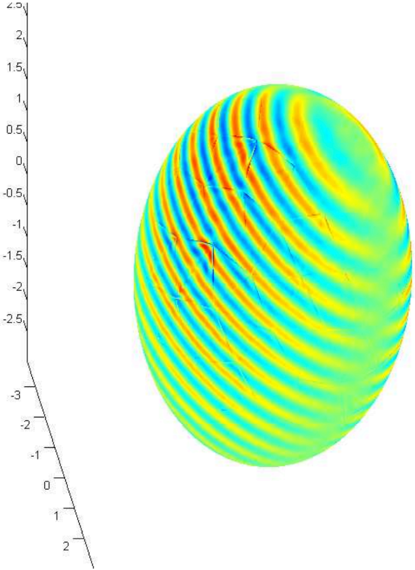


Non-Resonant Charge-Current Integral Equation

- $N_{triangles} = 116$
- $N_{unknowns} = 15660$
- $D = 9.5\lambda$
- Error $\mathbf{E}^{\mathbf{scat}} = 3.1 \cdot 10^{-3}$
- Error $\mathbf{H}^{\mathbf{scat}} = 1.4 \cdot 10^{-3}$



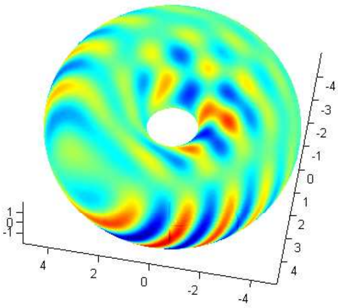
Numerical Experiments (High Frequency)

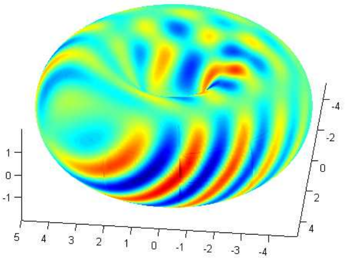


Non-Resonant Charge-Current Integral Equation

- $N_{triangles} = 236$
- $N_{unknowns} = 31860$
- $D = 19\lambda$
- Error $\mathbf{E}^{\mathbf{scat}} = 5.6 \cdot 10^{-2}$
- Error $\mathbf{H}^{\mathbf{scat}} = 2.2 \cdot 10^{-2}$
- Matrix filling time: 731 s
- Frontal solver: 578 s
  Intel(R) Xeon(R) CPU E5-2680 0 @ 2.7GHz, 64 GB RAM, 8 Cores

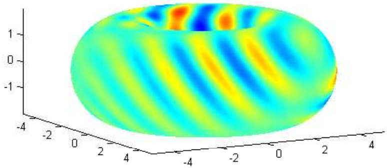
Numerical Experiments (High Frequency)

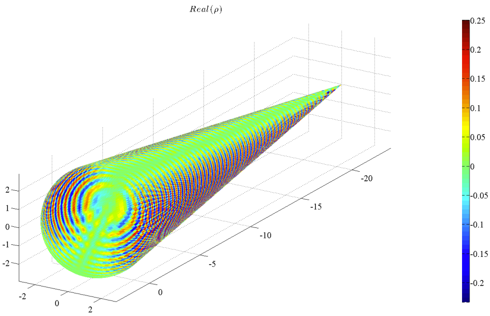




DPIE

- $N_{triangles} = 242$
- $N_{unknowns} = 16335$
- $D = 6.4\lambda$
- Error $\mathbf{E}^{\mathbf{scat}} = 3.9 \cdot 10^{-4}$
- Error $\mathbf{H}^{\mathbf{scat}} = 3.6 \cdot 10^{-4}$



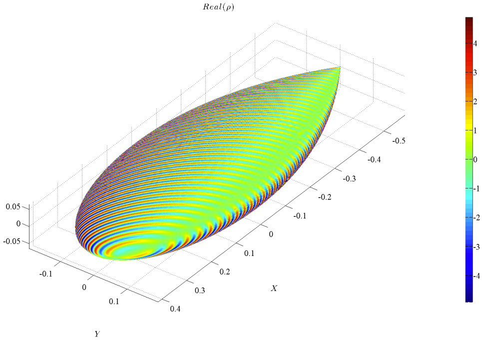


FMM. Gimbutas Greengard Library

Non-Resonant Charge-Current Integral Equation

- $N_{triangles} = 65535$
- $N_{unknowns} = 196605$
- $D_{max} = 70\lambda$
- Iterations 54
- CPU Time: 1h 13 min 43 sec
- Error $\mathbf{E}^{\mathbf{scat}} = 7.02 \cdot 10^{-2}$
- Error $\mathbf{H}^{\mathbf{scat}} = 4.92 \cdot 10^{-2}$

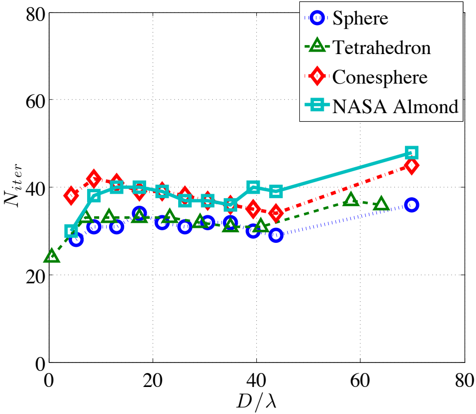
Numerical Experiments



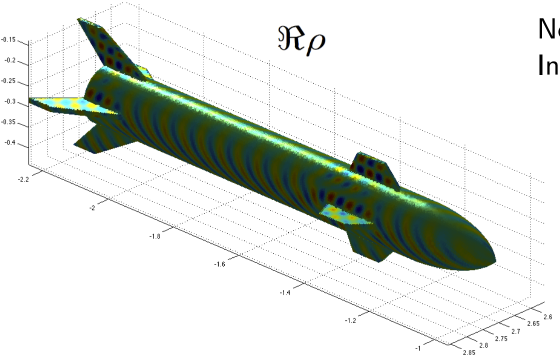
Non-Resonant Charge-Current Integral Equation

- $N_{triangles} = 327680$
- $N_{unknowns} = 983040$
- $D = 70\lambda$
- Iterations 48
- CPU Time: 9h 5 min 22 sec
- Error $\mathbf{E}^{\mathbf{scat}} = 2.93 \cdot 10^{-2}$
- Error $\mathbf{H}^{\mathbf{scat}} = 1.54 \cdot 10^{-2}$

Number of GMRES iterations:



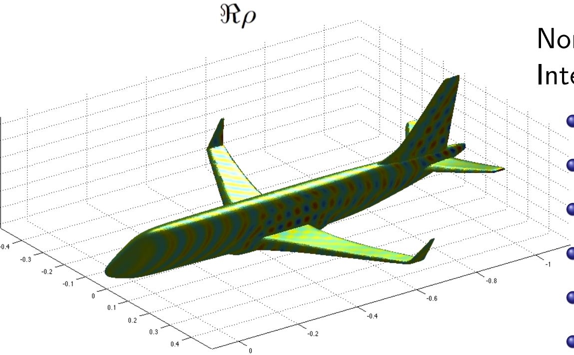
Rocket



Non-Resonant Charge-Current Integral Equation

- $N_{triangles} = 58562$
- $N_{unknowns} = 175686$
- $D_{max} = 36.82\lambda$
- Iterations 55
- Error $\mathbf{E}^{\mathbf{scat}} = 3.2 \cdot 10^{-2}$
- Error $\mathbf{H}^{\mathbf{scat}} = 2.9 \cdot 10^{-2}$

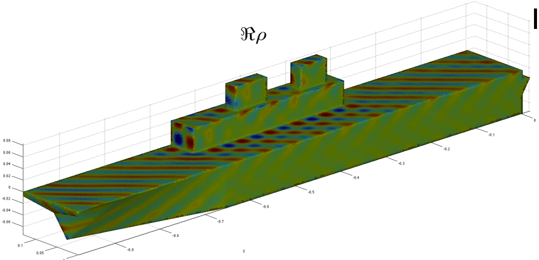
Plane



Non-Resonant Charge-Current Integral Equation

- $N_{triangles} = 122935$
- $N_{unknowns} = 368805$
- $D_{max} = 34.6\lambda$
- Iterations 59
- Error $\mathbf{E}^{\mathbf{scat}} = 2.5 \cdot 10^{-2}$
- Error $\mathbf{H}^{\mathbf{scat}} = 3.6 \cdot 10^{-2}$

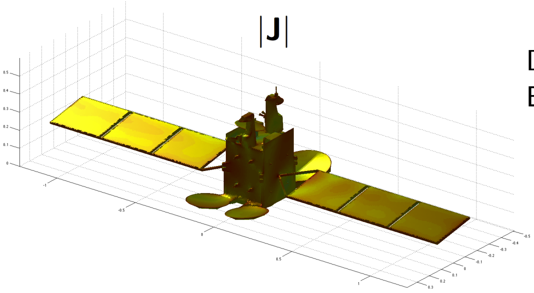
Carrier

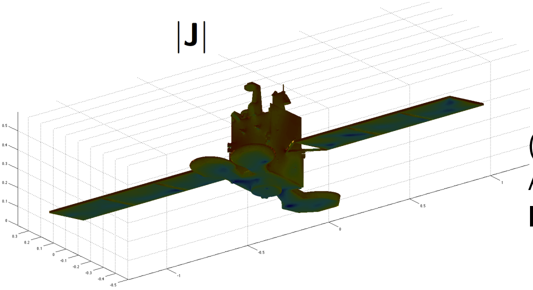


Non-Resonant Charge-Current Integral Equation

- $N_{triangles} = 58960$
- $N_{unknowns} = 176880$
- $D_{max} = 28.64\lambda$
- Iterations 52
- Error $\mathbf{E}^{\mathbf{scat}} = 7.1 \cdot 10^{-2}$
- Error $\mathbf{H}^{\mathbf{scat}} = 7.5 \cdot 10^{-2}$

Satellite. European Antenna Modelling Library. ESA





Decoupled Potential Integral Equation. FMM. First order

- $N_{triangles} = 55498$
- $N_{unknownsDPIEv} = 166495$
- $D_{max} = 0.1\lambda$
- Genus $= 6$
- Iterations 57
- Error $\mathbf{A}^{\mathbf{scat}} = 9.1 \cdot 10^{-2}$
- Error $\mathbf{H}^{\mathbf{scat}} = 6.4 \cdot 10^{-2}$

(Thanks to ESA European Antenna Modelling Library and IDS.)

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$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} w_k f(x_k)$$

Trapezoidal rule:

$$\int_a^b f(x)dx \approx \frac{(b-a)}{n}\Big(\frac{f(a)}{2} + \sum_{k=1}^{n-1} f(a+k(b-a)/n) + \frac{f(b)}{2}\Big)$$

$$Error = -\frac{(b-a)^2}{12n^2}(f'(b)-f'(a)) = O(n^{-2})$$

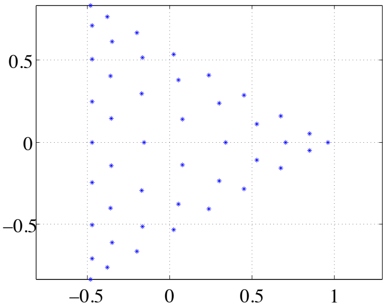
Gauss rule: (exact for polynomials of degree up to $2n-1$)

$$\int_a^b f(x)dx \approx \sum_{k=1}^{n} w_k f(x_k)$$

$$Error = -\frac{(b-a)^{2n+1}(n!)^4}{(2n+1)((2n)!)^3}(f^{2n}(\xi))$$

$$\int\int_{\Delta} f(\mathbf{x})dxdy \approx \sum_{k=1}^{n} w_k f(\mathbf{x}_k)$$

Gauss-like rule: (exact for polynomials of degree as high as possible)



Trapezoidal rule for periodic functions

High accuracy of Trapezoidal rule for periodic functions:

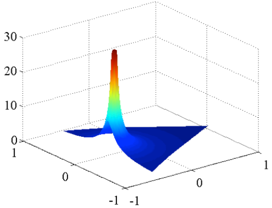
$$\int_0^1 f(x)dx \approx \frac{1}{n}\sum_{k=1}^{n} f(k/n)$$

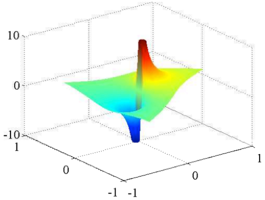
$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{i2\pi mx}$$

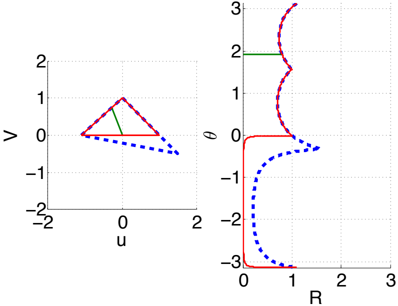
Trapezoidal quadrature rule exact for the first n terms of the Fourier series

$$Error \leq b_n n^{-k} \ \{b_n\} \in l^2 \Leftrightarrow f[0,1] - periodic \in W^k_{1,L^2}$$

[1] Rahman, Q. I., Schmeisser, G. (1990). Characterization of the speed of convergence of the trapezoidal rule. Numerische Mathematik, 57(1), 123-138.





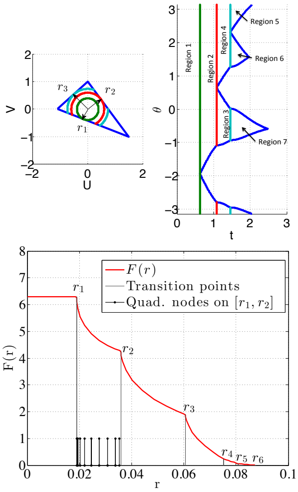


$$\int\int_{\Delta} \frac{1}{\sqrt{au^2 + bv^2}} f(u,v)\,du\,dv$$

Polar coordinates around the target point:

$$u = R\cos(\theta)$$
$$v = R\sin(\theta)$$

Quadratures of singular integrals



$$u = r(t)\cos(\theta)$$
$$v = r(t)\sin(\theta)$$

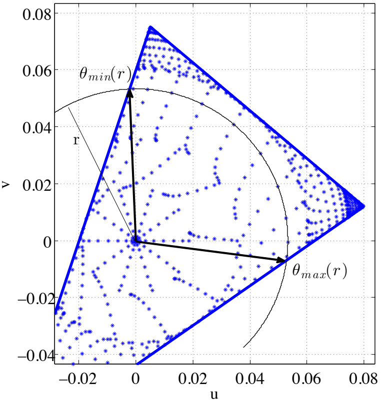
where $r(t)$ is defined piecewise as:

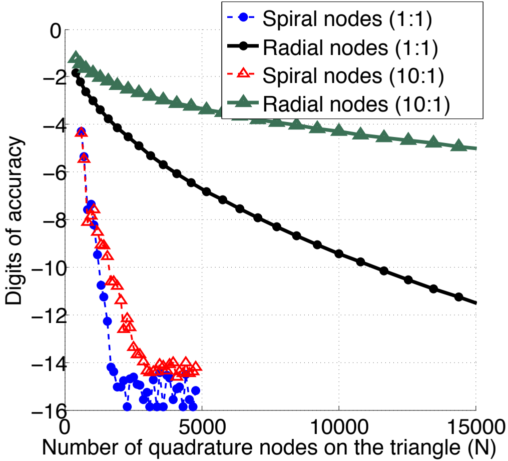
$$r(t) = t, \qquad t < t_1$$
$$r(t) = r_n + (t - t_n)^2, \quad t_n \leq t < t_{n+1}$$

where:

$$t_1 = r_1$$
$$t_{n+1} = t_n + \sqrt{r_{n+1} + r_n}$$

Quadratures of singular integrals



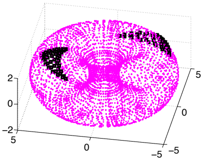


Integral transform. Fast evaluation

$$h(y) := \int_a^b g(x,y) f(x) dx$$

Gauss quadrature for each target point: (cost n^2)

$$h(y_j) \approx \sum_{i=1}^{n} g(x_i, y_j) f(x_i) w_i$$



$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \end{bmatrix} \begin{bmatrix} * & * & * \end{bmatrix} + \begin{bmatrix} * \\ * \\ * \end{bmatrix} \begin{bmatrix} * & * & * \end{bmatrix}$$

$$g(x,y) \approx c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 y^2 + c_5 xy + \ldots$$

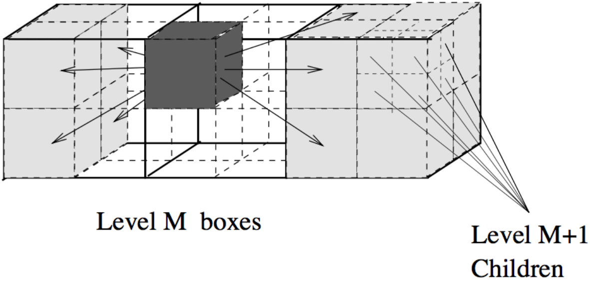
(cost np)

Integral transform. Fast evaluation. Singular kernels

$$g(\mathbf{x},\mathbf{y}) = log(|\mathbf{x}-\mathbf{y}|) \quad g(\mathbf{x},\mathbf{y}) = H_0^{(1)}(\lambda|\mathbf{x}-\mathbf{y}|)$$

$$g(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \qquad g(\mathbf{x},\mathbf{y}) = \frac{e^{i\lambda|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$$

Fast Multipole Method: Tree structure. Previous method on well-separated boxes. (cost nlog(n))



[2] Greengard, L., Rokhlin, V. (1987). A fast algorithm for particle simulations. Journal of computational physics, 73(2), 325-348.

GMRES

Iterative methods to solve linear systems: Krylow subspace

$$f(\mathbf{y}) + \int g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}) d\mu_{\mathbf{x}} = h(\mathbf{y})$$

$$\{\delta_{ij} + g_{ij} w_i\}_{ij=1}^{n} \{f_i\}_{i=1}^{n} = \{h_j\}_{j=1}^{n}$$

$$Ax = b$$

$$K_m = K_m(A, b) = span\{b, Ab, A^2 b, ..., A^{m-1} b\}$$

$$x_m \in K_m$$

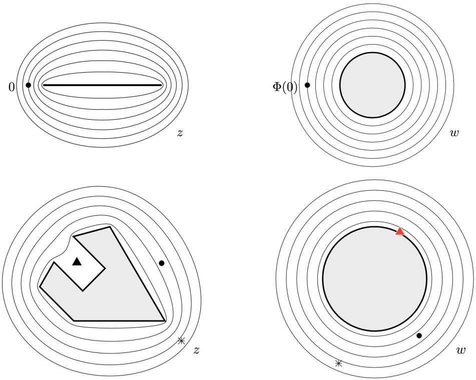
$$r_m = Ax_m - b$$

$$\frac{|r_m|}{|r_0|} \approx \rho^n$$

$$|r_m| \leq \inf_{p \in P_m} |p(A)|, |r_m| \leq \left(\frac{\kappa_2(A)^2 - 1}{\kappa_2(A)^2}\right)^{m/2} |r_0|$$

Integral transform. Fast evaluation. Singular kernels

$$\rho = \frac{1}{|\Phi(0)|}$$



[3] Driscoll, T. A., Toh, K. C., Trefethen, L. N. (1998). From potential theory to matrix iterations in six steps. SIAM review, 40(3), 547-578.