Return times and synchronous recurrence

Piotr Oprocha

Faculty of Applied Mathematics
AGH University of Science and Technology, Kraków, Poland

May 8, 2014, UPV, Valencia, Spain

Dynamical system

1. X – always denotes a compact metric space (X, d).
2. $f : X \to X$ – always a continuous map.
3. as usual $\quad f^0(x) = x, \quad f^{n+1}(x) = f(f^n(x))$.
4. $O_f(x) = \{x, f(x), f^2(x), \dots\}$ – orbit of a point x.
5. Sets of transfer times:
    1. $N_f(U, V) = \{n : f^n(U) \cap V\} \neq \emptyset$.
    2. $N_f(x, V) = \{n : f^n(x) \in V\}$.

Recurrence and uniform recurrence

1. x is recurrent if $N_f(x,U) \neq \emptyset$ for any open $U \ni x$
2. x is **uniformly** recurrent if additionally $N_f(x,U)$ is syndetic, i.e. there is $k > 0$ such that

$$N_f(x,U) \cap [i, i+k] \neq \emptyset \quad \text{for every } i \geq 0.$$

3. x is uniformly recurrent iff $\overline{O_f(x)}$ is a minimal set, i.e. there is no proper subset $M \subsetneq \overline{O_f(x)}$ such that
   1. M is closed
   2. $M \neq \emptyset$,
   3. $f(M) \subset M$.

Product recurrence

Hillel Furstenberg
(Oberwolfach – 1972)

1. $x \in X$ is product recurrent if
   1. given any recurrent point y
   2. in any dynamical system (Y, g)
   3. and any open neighborhoods U of x and V of y,
   4. $N_f(x, U) \cap N_g(y, V) \neq \emptyset$.

Full characterization - Furstenberg's theorem

1. $x, y \in X$ are proximal if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.
2. x is distal if it is not proximal to any $y \in \overline{O_f(x)}$, $y \neq x$.
3. (X, f) is distal if every $x \in X$ is distal.

Theorem

If x is distal then x is uniformly recurrent (i.e. $\overline{O_f(x)}$ is a minimal set).

Theorem (Furstenberg)

A point x is product recurrent if and only if it is (uniformly recurrent) distal point.

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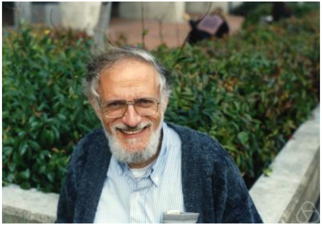
Weak product recurrence – problem revisited

Joseph Auslander



Hillel Furstenberg

1. $x \in X$ is **weakly** product recurrent if
   1. given any **uniformly** recurrent point y in any dynamical system (Y, g)
   2. and any neighborhoods U of x and V of y,
   3. $N_f(x, U) \cap N_g(y, V) \neq \emptyset$.

Question

,,Another question (even for $\mathbb{Z}$ or $\mathbb{N}$ actions): If (x, y) is recurrent for all uniformly recurrent points y, is x necessarily a distal point?"
[J. Auslander and H. Furstenberg, *Product recurrence and distal points*, Trans. Amer. Math. Soc., **343** (1994) 221–232.]

A theorem by Haddad and Ott [ETDS, 2008]

Corollary

A point $x \in X$ is weakly product recurrent if the following conditions hold:

1. The orbit of x is dense in X.
2. For any neighborhood V of x there exists N such that for any $k \in \mathbb{N}$, if $n_i \geq N$ for $1 \leq i \leq k$, then the intersection

$$V \cap f^{-n_1}(V) \cap \ldots \cap f^{-(n_1+\ldots+n_k)}(V) \neq \emptyset.$$

If f is topologically exact, i.e.

- for every nonempty open set U
- there is n such that $f^n(U) = X$

then assumptions of the above theorem are fulfilled.

Question (Haddad & Ott)

If x is uniformly recurrent and weakly product recurrent, is it necessarily distal?

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Other types of mixing

1. f is transitive if for every nonempty open set U, V there is n such that $f^n(U) \cap V \neq \emptyset$.
2. f is (topologically) weakly mixing if $f \times f$ is transitive. Equivalently: $f^n(U_i) \cap V_i \neq \emptyset$ for $i = 1, \ldots, m$, $n, m > 0$.
3. f is (topologically) mixing if there is N such that $f^n(U) \cap V \neq \emptyset$ for $n > N$.

$$\text{Exact} \implies \text{Mix} \implies \text{WMix} \implies \text{Trans}$$

4. We say that f has small periodic sets if for any nonempty open set U there are a point $x \in U$ and an integer $K > 0$ such that

$$f^{nK}(x) \in U \quad \text{for all } n \in \mathbb{N}.$$

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Disjointness

1. A closed set $\emptyset \neq J \subset X \times Y$ is a joining of (X, f) and (Y, g) if it is invariant (for the product map $f \times g$) and its projections on first and second coordinate are X and Y respectively.
2. If $X \times Y$ is the only joining of f and g then we say that f and g are disjoint.

Question (Furstenberg)

How to characterize systems disjoint from any distal or minimal system? [H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory, **1** (1967), 1–49]

Theorem (Petersen, 1970)

A system is disjoint with every distal system iff it is weakly mixing and minimal.

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More on disjointness

Theorem (Furstenberg, 1967)

If f is weakly mixing with dense periodic points then it is disjoint from all minimal systems.

Theorem (Huang & Ye)

If f is weakly mixing with small periodic sets then it is disjoint from minimal systems.

Theorem (Huang & Ye; O.)

If f is weakly mixing with dense set of distal points then it is disjoint from all minimal systems.

Theorem (Huang & Ye; O.)

If f is disjoint with any minimal system then every point with dense orbit is weakly product recurrent.

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A few standard (Furstenberg) families

1. $\mathscr{F} \subset \mathbb{N}$ – family if hereditary upward (i.e. $A \subset B$, $A \in \mathscr{F}$ then $B \in \mathscr{F}$).
2. $\mathscr{F}_{inf}$ – infinite subsets of $\mathbb{N}$,
3. $\mathscr{F}_s$ – syndetic subsets,
4. $\mathscr{F}_t$ – thick subsets, i.e. complements of syndetic sets (contain arbitrary long blocks of consecutive integers),
5. $\mathscr{F}_{ps}$ – piecewise syndetic sets, i.e. $F \in \mathscr{F}_{ps}$ if $F = A \cap B$, $A \in \mathscr{F}_s$, $B \in F_t$.
6. $\mathscr{F}_{pubd}$ – sets with positive upper Banach density, that is sets $F \subset \mathbb{N}$ such that
$$\limsup_{n-m\to\infty} \frac{\#(F \cap \{m, m+1, \cdots, n\})}{n-m+1} > 0,$$

Generalized product recurrence
Weaker recurrence (Dong, Shao, Ye, **2010?**)

1. x is $\mathscr{F}$-recurrent if $N_f(x, U) \in \mathscr{F}$ for any open neighborhood U of x.
2. It is clear that:
   - x recurrent iff it is $\mathscr{F}_{inf}$-recurrent,
   - x is uniformly recurrent iff it is $\mathscr{F}_s$-recurrent.
3. A recurrent point x in a dynamical system (X, f) is $\mathscr{F}$-product recurrent ($\mathscr{F}$-PR for short) if
   - for any $\mathscr{F}$-recurrent point y
   - in any dynamical system (Y, g),
   - the pair (x, y) is recurrent for $(X \times Y, f \times g)$.
4. Again, it is clear that:
   - x product recurrent iff it is $\mathscr{F}_{inf}$-PR,
   - x is weakly product recurrent iff it is $\mathscr{F}_s$-PR.

[P. Dong, S. Shao, X. Ye, *Product recurrent properties, disjointness and weak disjointness*, Israel J. Math., 188 (2012), 463–507]

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Generalized product recurrence: Smaller class of systems (Dong, Shao, Ye, 2012)

1. A recurrent point x in a dynamical system (X, f) is $\mathscr{F} - \mathrm{PR}_0$ if
   - for any $\mathscr{F}$-recurrent point y
   - in any dynamical system (Y, g) with zero topological entropy,
   - the pair (x, y) is recurrent for $(X \times Y, f \times g)$.

Simple observations

1. $\mathscr{F}_s \subset \mathscr{F}_{ps} \subset \mathscr{F}_{pubd} \subset \mathscr{F}_{inf}$.
2. If x is $\mathscr{F}_{\mathrm{pubd}}$-recurrent then its orbit closure is a transitive system with a fully supported measure.

Why to choose exactly these families?

Theorem (Dong, Shao, Ye)

If x is a transitive point in a system disjoint from all minimal zero entropy systems then it is $\mathscr{F}_s - \mathrm{PR}_0$

Theorem (Dong, Shao, Ye)

A point $x \in X$ is distal iff it is $\mathscr{F}_{\mathrm{inf}}$-$\mathrm{PR}_0$.

Theorem (Dong, Shao, Ye)

If x is $\mathscr{F}_{\mathrm{ps}}$-$\mathrm{PR}_0$ then it is a uniformly recurrent point. In particular, each $\mathscr{F}_{ps}$-PR point is uniformly recurrent.

Collecting more facts on disjointness (Dong, Shao, Ye)

Theorem (Huang, Park, Ye)

If T is a homeomorphism such that any of its invariant measures is K-measure then it is disjoint with any zero entropy system with fully supported measure.

Remark

If x is $\mathscr{F}_{\text{pubd}}$-recurrent then its orbit closure $\overline{O^+(x)}$ is system with fully supported measure.

Corollary

If x is a transitive point in strictly ergodic K-system then x is $\mathscr{F}_{\text{pubd}} - \text{PR}_0$

1. $\mathscr{F}_{\text{pubd}} - \text{PR}_0$ point does not have to be distal,
2. $\mathscr{F}_{\text{pubd}} - \text{PR}_0$ point does not have to be $\mathscr{F}_s$-PR (take an asymptotic pair (x, y) in minimal K-system; it is not recurrent).

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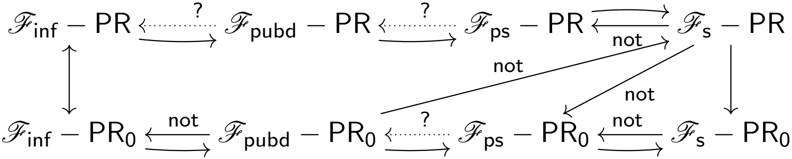
Relations between different PR properties



Theorem (O. & Zhang)

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- [O. & G.H. Zhang, *On weak product recurrence and synchronization of return times*, Adv. Math., **244** (2013), 395–412].

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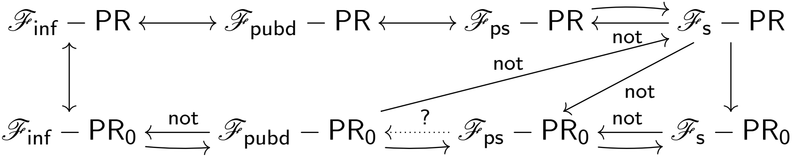

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