Hereditarily indecomposable continua and entropy.
(joint work with Jan P. Boroński and Alex Clark)

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Indecomposable arc-like continua

1. Continuum $C$ is **arc-like** if for every $\varepsilon > 0$ there is an $\varepsilon$-map $\pi: C \to I$ (i.e. $\text{diam} \pi^{-1}(x) < \varepsilon$ for every $x \in I$).
2. **Circle-like**, **tree-like**, **graph-like** are defined analogously.

3. Continuum $C$ is **indecomposable** if it is not the union of two proper subcontinua.
4. **Hereditarily indecomposable** if all nondegenerate subcontinua are indecomposable.
5. Arc-like hereditarily indecomposable continuum is topologically unique - we call it the **pseudoarc** (Knaster; Moise; Bing).
(1951) R.H. Bing: pseudo-circle, a hereditarily indecomposable circle-like continuum that separates the plane into exactly two components. (uniqueness! Fearnley&Rogers)

Figure: Construction by crooked circular chain (picture by Charatonik&Prajs&Pyrih)
We call an arc $A$ $\epsilon$-crooked if for any pair of points $p$ and $q$ in $A$ there exist points $r$ and $s$ between $p$ and $q$ such that $r$ is between $p$ and $s$, $|p - s| < \epsilon$, and $|r - q| < \epsilon$.

**Figure:** $\epsilon$-crooked arc from $p$ to $q$
(1960) Fearnley & Rogers: pseudo-circle is not homogeneous
(1986) Kennedy & Rogers: pseudo-circle is uncountably nonhomogeneous
(2011) Sturm: pseudo-circle is not continuously homogeneous

(1991) Bellamy & Lewis: the two-point compactification of the universal cover of the pseudo-circle is the pseudo-arc
M. Brown (1958): There exists a continuous decomposition of \( \mathbb{R}^2 \setminus \{(0,0)\} \) into pseudo-circles.
(1982) Handel: pseudo-circle as an attracting minimal set of a $C^\infty$-smooth diffeomorphism of the plane

(1996) Kennedy&Yorke: constructed a $C^\infty$ diffeomorphism on a 7-manifold which has an invariant set with an uncountable number of pseudocircle components and is stable to $C^1$ perturbations

(2010) Chéritat: pseudo-circle as a boundary of a Siegel disk of a holomorphic map
1. In general, inverse limit -
\[ \lim\left\{ \{f_i\}_{i=0}^{\infty}, X \right\} = \{(x_0, x_1, \ldots) : x_i \in X, f_i(x_{i+1}) = x_i\} \]

2. We are interested in cases when there is one bonding map:
\[ \mathbb{X} = \lim\{f, X\} = \{(x_0, x_1, \ldots) : x_i \in X, f(x_{i+1}) = x_i\} \]

3. Shift homeo. - \( \sigma_f(x_0, x_1, \ldots) = (f(x_0), x_0, x_1, \ldots) \)

4. \( f \) and \( \sigma_f \) share many dynamical properties, e.g.
   - Dense periodic points
   - Admissible periods of periodic points
   - \( h_{\text{top}}(f) = h_{\text{top}}(\sigma_f) \)
   - .....
If $f \in C(I)$ has some special properties, then $X$ is a pseudoarc.

Then we can study dynamical properties of the homeomorphism $\sigma_f$ in terms of $f$.

Method of Minc and Transue

Theorem (Kawamura, Tuncali & Tymchatyn)
Let $G$ be a topological graph and $f : G \to G$ a piecewise linear surjection which satisfies the following condition (topological exactness):

- for each open subset $U$ of $G$, there is a positive integer $n$ such that $f^n(U) = G$.

Then for each $\varepsilon > 0$ there is a map $f_\varepsilon : G \to G$ which is $\varepsilon$-close to $f$ such that $(G, f_\varepsilon)$ is hereditarily indecomposable.

Theorem (Kościelniak & O. & Tuncali)
Let $G$ be a topological graph and let $\mathcal{K}$ be a triangulation of $G$. For every topologically exact map $f : G \to G$ and every $\varepsilon > 0$ there is a topologically mixing map $f_\varepsilon : G \to G$ with the shadowing property, which is $\varepsilon$-close to $f$ such that $(G, f_\varepsilon)$ is hereditarily indecomposable and $f(x) = f_\varepsilon(x)$ for every vertex in $\mathcal{K}$. 
Theorem (Barge & Martin)

Every continuum $X = \lim \{ f, [0, 1] \}$, can be embedded into a disk $D$ in such a way that

(i) $X$ is an attractor of a homeomorphism $h: D \to D$,

(ii) $h|_X = \sigma_f$; i.e. $h$ restricted to $X$ agrees with the shift homeomorphism induced by $f$, and

(iii) $h$ is the identity on the boundary of $D$.

Remark

It was pointed by Barge & Roe that the same is true if $f$ is a degree $\pm 1$ circle map and $h$ is an annulus homeomorphism.
Degree one map
Embedding
Pseudo-circle as strange attractor

**Theorem (Barge & Gillette, 1991)**

Suppose $h: A \to A$ is an orientation preserving annulus homeomorphism with an invariant cofrontier $C$. If the rotation number of $h|_C$ is not unique then $C$ is indecomposable, the set of rotation numbers contains an interval, and each rational rotation number is realized by a periodic orbit.

**Theorem (Boroński & O., 2015)**

There exists a 2-torus homeomorphism $h$ homotopic to identity such that:
- a pseudocircle $C$ is a strange attractor for $h$ (i.e. set of rotation numbers on $C$ contains an interval),
- $T|_C$ is topologically mixing and has positive entropy,
- the set of rotation numbers of $T|_C$ is an interval.
An old question

**Question 1 (Barge, 1989?)**
Is every real number the entropy of some homeomorphism on the pseudo-arc?

**Theorem (Mouron, 2012)**
If $f \in C(I)$ is such that the inverse limit $\mathbb{X}$ is the pseudoarc then $h_{\text{top}}(f) \in \{0, \infty\}$.

- The answer to Barge’s question is still unknown.
- With Example of Henderson + Minc and Transue technique we see that both cases $0, \infty$ can be obtained in practice.
A related result

- We proved the following (with other methodology than Mouron).

**Theorem (Boroński & O.)**

If \( f \in C(G) \) is such that the inverse limit \( \mathbb{X} \) is the **hereditarily indecomposable** then:

1. \( h_{\text{top}}(f) \in \{0, \infty\} \),

- it is known that there is a homeomorphism of the pseudo-circle with zero entropy - example of M. Handel.
On the circle

**Theorem (Boroński & O.)**

If $X = \lim(S^1, f)$ is hereditarily indecomposable, and $X$ is not the pseudo-arc (i.e. $\deg f \neq 0$) then $h_{\text{top}}(f) = \infty$.

- Main ingredient is characterization of Auslander and Katznelson of periodic-point-free circle homeomorphisms,
- We believe that $h_{\text{top}}(f) = \infty$ when $G$ has no endpoints, but have no proof so far...

**Corollary**

Suppose $f : S^1 \to S^1$ is a map with $\deg(f) = 1$. If $\Lambda_f = \lim(S^1, f)$ is hereditarily indecomposable then the rotation set $\rho(f)$ (after embedding by BBM) is nondegenerate.
On the circle

**Theorem (Boroński & O.)**

If \( X = \lim(\mathbb{S}^1, f) \) is hereditarily indecomposable, and \( X \) is not the pseudo-arc (i.e. \( \text{deg} f \neq 0 \)) then \( h_{\text{top}}(f) = \infty \)

- Main ingredient is characterization of Auslander and Katznelson of periodic-point-free circle homeomorphisms,
- We believe that \( h_{\text{top}}(f) = \infty \) when \( G \) has no endpoints, but have no proof so far...

**Corollary**

Suppose \( f : \mathbb{S}^1 \to \mathbb{S}^1 \) is a map with \( \text{deg}(f) = 1 \). If \( \Lambda f = \lim(\mathbb{S}^1, f) \) is hereditarily indecomposable then the rotation set \( \rho(f) \) (after embedding by BBM) is nondegenerate.
(1982) M. Handel: pseudo-circle as minimal set and attractor

Figure: by M. Handel
Theorem (Handel, 1982)

There exists a $C^\infty$-smooth diffeomorphism of the plane $F$ with pseudo-circle as an attracting minimal set. In addition, $F$ has a well defined irrational rotation number but is not semi-conjugate to a circle rotation (Thurston?).

Question 2 (Auslander, AIMS Madrid 2014)

Does Handel’s homeomorphism has any proximal pairs?

- A pair of points $x, y$ is **proximal** for $h$ if
  \[ \liminf_{n \to \infty} d(h^n(x), h^n(y)) = 0. \]
Dynamical properties of Handel’s construction

1. Handel’s example is (implicitly contained in Handel’s paper)
   - minimal
   - uniformly rigid

Observation (Boroński, Clark, O.)
Handel’s example is weakly mixing (in particular, proximal pairs form a residual set).
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Observation (Boroński, Clark, O.)
Handel’s example is weakly mixing (in particular, proximal pairs form a residual set).
Examples of minimal, weakly mixing and uniformly rigid systems were first constructed in dimension 2 or higher by Glassner and Maon in 1989.

These examples include $\mathbb{T}^n$ for every $n \geq 2$.

Dynamical properties can be slightly extended beyond topological weak mixing (e.g. weakly mixing measure)

or on other surfaces (e.g. Klein bottle; example by K. Yancey)

Up to our knowledge, this is the only second class of such examples, and the first in dimension 1.
Recall:

**Question 1 (Barge, 1989)**
Is every real number the entropy of some homeomorphism on the pseudo-arc?

**Question 3 (Mouron 2009)**
Does there exist a homeomorphism of a hereditarily indecomposable continuum with topological entropy other than 0 or $\infty$?
A solution...

Theorem (Boronski & Clark & O.)

There exist hereditarily indecomposable continua $X$ satisfying:

1. there is $\alpha \in (0, 1)$ and a homeomorphism $T_\alpha : X \to X$ with $h_{top}(T_\alpha) = \alpha$ (in fact there are infinitely many such $\alpha$)

2. $X$ occurs as an invariant minimal set with intermediate complexity within an attractor of a smooth diffeomorphism $F$ of a 4 dimensional manifold, and

3. the restriction $F|X$ is weakly mixing.
A brief look on formal methodology...

**Theorem (Boronski & Clark & O.)**

Assume that $H$ is a HAK homeomorphism (e.g. the one from Handel’s construction) with a nonzero rotation number $\alpha$ on pseudo-circle $\Psi$. Then $h_{\text{top}}(HC) = |\alpha|h_{\text{top}}(h)$.

$HC$ “lives” in the quotient space $([0, 1] \times \mathbb{R}) \times C / \approx$, where:

- $h : C \to C$ is a minimal homeomorphism,
- $((s, r), c) \approx ((s', r'), c')$ if and only if
- $s = s'$ and there is an $n \in \mathbb{Z}$ satisfying $r' = r + n$ and $c' = h^{-n}(c)$,
\[ H \]
$C$ \rightarrow \mathcal{H}$
Some further questions

**Question 4a**
Does for every $\alpha \in \mathbb{R}$ there exist a pseudo-circle homeomorphism with a well defined rotation number $\alpha$?

**Question 4b**
Is there a hereditarily indecomposable continuum $X$ such that for every $t \geq 0$ there is a homomorphism $F_t : X \to X$ of entropy $h_{\text{top}}(F_t) = t$?

Note: See a series of papers by John Mayer, and a paper by Mark Turpin.

**Question 5**
Is there an indecomposable cofrontier that admits a minimal homeomorphism semi-conjugated to an irrational circle rotation?
Theorem (Boroński & Clark & O.)

Let $\Psi \subset A$ be an essential pseudo-circle attracting all the points from int $A$ and assume that $H : A \to A$ is a homemomorphism with a nondegenerate rotation set. Then $h_{top}(H|\Psi) = h_{top}(H) = +\infty$.

Question 5

Let $\Psi \subset A$ be an essential pseudo-circle and assume that $H : A \to A$ is a homemomorphism with a nondegenerate rotation set. Is it true that $h_{top}(H|\Psi) = +\infty$. 
That’s all...

Thank you for your attention.

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New exotic minimal sets from pseudo-suspensions of Cantor systems