On non-completable fuzzy metric spaces

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(George and Veeramani [1].) A fuzzy metric space is an ordered triple $(X, M, *)$ such that $X$ is a (non-empty) set, $*$ is a continuous $t$-norm and $M$ is a fuzzy set on $X \times X \times ]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

1. **(GV1)** $M(x, y, t) > 0$;
2. **(GV2)** $M(x, y, t) = 1$ if and only if $x = y$;
3. **(GV3)** $M(x, y, t) = M(y, x, t)$;
4. **(GV4)** $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
5. **(GV5)** $M(x, y, t) : [0, \infty[ \rightarrow [0, 1]$ is continuous.

If axioms (GV1), (GV2) and (GV5) are replaced by

1. **(KM1)** $M(x, y, 0) = 0$
2. **(KM2)** $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$
3. **(KM5)** $M(x, y, t) : [0, \infty[ \rightarrow [0, 1]$ is left continuous

respectively, we obtain the concept of $KM$-fuzzy metric space.
Let \((X, d)\) be a metric space and let \(M_d\) a fuzzy set on \(X \times X \times ]0, \infty[\) defined by
\[
M_d(x, y, t) = \frac{t}{t + d(x, y)}
\]
Then \((X, M_d, \cdot)\) is a fuzzy metric space and \(M_d\) is called the standard fuzzy metric induced by \(d\).

**Definition (Gregori and Romaguera [9].)**

A fuzzy metric \(M\) on \(X\) is said to be **stationary** if \(M\) does not depend on \(t\), i.e. if for each \(x, y \in X\), the function \(M_{x,y}(t) = M(x, y, t)\) is constant. In this case we write \(M(x, y)\) instead of \(M(x, y, t)\).

\(M\) generates a topology \(\tau_M\) on \(X\) which has a base the family of open sets of the form \(\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}\), where \(B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}\) for all \(x \in X, \epsilon \in ]0, 1[\) and \(t > 0\).
Proposition

(George and Veeramani [1]). A sequence $(x_n)_n$ in $X$ converges to $x$ if and only if $\lim_n M(x_n, x, t) = 1$, for all $t > 0$.

Definitions (George and Veeramani [1].)

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \ast)$ is said to be M-Cauchy, or simply Cauchy, if for each $\epsilon \in ]0, 1]$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$, or equivalently, if $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$.

$X$ is said to be complete if every Cauchy sequence in $X$ is convergent with respect to $\tau_M$. In such a case $M$ is also said to be complete.
Definition (Gregori and Romaguera [8].)

Let \((X, M, \ast)\) and \((Y, N, \lozenge)\) be two fuzzy metric spaces. A mapping \(f\) from \(X\) to \(Y\) is said to be an **isometry** if for each \(x, y \in X\) and \(t > 0\), 
\[ M(x, y, t) = N(f(x), f(y), t) \]
and, in this case, if \(f\) is a bijection, \(X\) and \(Y\) are called **isometric**.

A **fuzzy metric completion** of \((X, M)\) is a complete fuzzy metric space \((X^*, M^*)\) such that \(X\) is isometric to a dense subspace of \(X^*\).

\(X\) is said to be **completable** if it admits a fuzzy metric completion.

Proposition (Gregori and Romaguera [8].)

If a fuzzy metric space has a fuzzy metric completion then it is unique up to isometry.
Theorem (Gregori and Romaguera [9].)

A fuzzy metric space \((X, M, \ast)\) is completable if and only if it satisfies the following conditions:

(C1) Given two Cauchy sequences \(\{a_n\}\) and \(\{b_n\}\) in \(X\), then 
\[
\lim_{n} M(a_n, b_n, s) = 1 \text{ for some } s > 0 \text{ implies } \lim_{n} M(a_n, b_n, t) = 1 \text{ for all } t > 0.
\]

(C2) Given two Cauchy sequences \(\{a_n\}\) and \(\{b_n\}\) in \(X\), then the assignment 
\[
t \rightarrow \lim_{n} M(a_n, b_n, t)
\]

is a continuous function on \(]0, \infty[\) with values in \([0, 1]\).
Example (Gregori and Romaguera [9].)

Let \( \{ x_n \} \) and \( \{ y_n \} \) be two strictly increasing sequences of positive real numbers, which converge to 1 with respect to the usual topology of \( \mathbb{R} \), with \( A \cap B = \emptyset \)
where \( A = \{ x_n : n \in \mathbb{N} \} \) and \( B = \{ y_n : n \in \mathbb{N} \} \). Put \( X = A \cup B \) and define a fuzzy set \( M \) on \( X \times X \times ]0, \infty[ \) by:

\[
M(x_n, x_n, t) = M(y_n, y_n, t) = 1 \quad \text{for all } n \in \mathbb{N}, t > 0,
M(x_n, x_m, t) = x_n \land x_m \quad \text{for all } n, m \in \mathbb{N} \text{ with } n \neq m, t > 0,
M(y_n, y_m, t) = y_n \land y_m \quad \text{for all } n, m \in \mathbb{N} \text{ with } n \neq m, t > 0,
M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \land y_m \quad \text{for all } n, m \in \mathbb{N}, t \geq 1,
M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \land y_m \land t \quad \text{for all } n, m \in \mathbb{N}, t \in ]0, 1[.
\]

\((X, M, \star)\) is a fuzzy metric space, where \( \star \) is the minimum \( t \)-norm.

The authors observed that \( M \) satisfies condition \((C2)\), but it does not satisfy condition \((C1)\). Indeed, in [9] was observed that \( \{ x_n \} \) and \( \{ y_n \} \) are Cauchy sequences in \( X \) such that \( \lim_{n} M(x_n, y_n, t) = 1 \) for all \( t \geq 1 \), but \( \lim_{n} M(x_n, y_n, t) = t \) for all \( t \in ]0, 1[\).
Example (Gregori and Romaguera [8].)

Let \( \{x_n\} \) and \( \{y_n\} \) be two sequences of distinct points such that \( A \cap B = \emptyset \), where \( A = \{x_n : n \geq 3\} \) and \( B = \{y_n : n \geq 3\} \). Put \( X = A \cup B \) and define a fuzzy set \( M \) on \( X \times X \times ]0, \infty[ \) by:

\[
M(x_n, x_m, t) = M(y_n, y_m, t) = 1 - \left[ \frac{1}{n \wedge m} - \frac{1}{n \vee m} \right],
\]

\[
M(x_n, y_m, t) = M(y_m, x_n, t) = \frac{1}{n} + \frac{1}{m},
\]

for all \( n, m \geq 3 \).

\((X, M, \ast)\) is a fuzzy metric space, where \( \ast \) is the Luckasievicz \( t \)-norm \( (a \ast b = \max\{0, a + b - 1\}) \).

\( M \) is stationary, and so it satisfies condition \((C1)\). However, \( M \) does not satisfy condition \((C2)\), since \( \{x_n\}_{n \geq 3} \) and \( \{y_n\}_{n \geq 3} \) are Cauchy sequences. Then

\[
\lim_n M(x_n, y_n, t) = \lim_n \left( \frac{1}{n} + \frac{1}{n} \right) = 0.
\]
Problem

To find a fuzzy metric space \((X, M, \ast)\) where for two \(M\)-Cauchy sequences \((a_n)_n\) and \((b_n)_n\) in \(X\) the assignment

\[ t \mapsto \lim_{n} M(a_n, b_n, t) \text{ for all } t > 0, \]

does not define a continuous function on \(t\), for the usual topology of \(\mathbb{R}\).
Let $d$ be the usual metric on $\mathbb{R}$ restricted to $]0, 1]$ and consider the standard fuzzy metric $M_d$ induced by $d$. Put $X = ]0, 1]$ and define a fuzzy set $M$ on $X \times X \times ]0, \infty[$ by

$$M(x, y, t) = \begin{cases} M_d(x, y, t), & 0 < t \leq d(x, y) \\ M_d(x, y, 2t) \cdot \frac{t-d(x, y)}{1-d(x, y)} + M_d(x, y, t) \cdot \frac{1-t}{1-d(x, y)}, & d(x, y) < t \leq 1 \\ M_d(x, y, 2t), & t > 1 \end{cases}$$

$(X, M, \cdot)$ is a fuzzy metric space.

Let $\{a_n\}$ and $\{b_n\}$, given by $a_n = \frac{1}{n}$ and $b_n = 1$ for all $n \in \mathbb{N}$. These are Cauchy sequences in $(X, M, \cdot)$ and the assignment

$$\lim_n M(a_n, b_n, t) = \begin{cases} \frac{t}{t+1}, & 0 < t < 1 \\ \frac{2t}{2t+1}, & t \geq 1 \end{cases}$$

is a well-defined function on $]0, \infty[$ which is not continuous at $t = 1$. 
A fuzzy metric space \((X, M, \star)\) is completable if and only if for each pair of Cauchy sequences \(\{a_n\}\) and \(\{b_n\}\) in \(X\) the following three conditions are fulfilled:

\[
\begin{align*}
\text{(c1)} & \quad \lim_{n} M(a_n, b_n, s) = 1 \text{ for some } s > 0 \text{ implies } \\
& \quad \lim_{n} M(a_n, b_n, t) = 1 \text{ for all } t > 0. \\
\text{(c2)} & \quad \lim_{n} M(a_n, b_n, t) > 0 \text{ for all } t > 0. \\
\text{(c3)} & \quad \text{The assignment } t \to \lim_{n} M(a_n, b_n, t) \text{ for each } t > 0 \text{ is a continuous function on } ]0, \infty[, \text{ provided with the usual topology of } \mathbb{R}.
\end{align*}
\]


