On non-completable fuzzy metric spaces

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Definition

(George and Veeramani [1].) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;

(GV2) $M(x, y, t) = 1$ if and only if $x = y$;

(GV3) $M(x, y, t) = M(y, x, t)$;

(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

If axioms (GV1), (GV2) and (GV5) are replaced by

(KM1) $M(x, y, 0) = 0$

(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$

(KM5) $M(x, y, \_) : [0, \infty[\to [0, 1]$ is left continuous

respectively, we obtain the concept of KM-fuzzy metric space.

Let (X, d) be a metric space and let M_d a fuzzy set on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space and M_d is called the standard fuzzy metric induced by d.

Definition (Gregori and Romaguera [9].)

A fuzzy metric M on X is said to be stationary if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

M generates a topology τ_M on X which has a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where
$B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$.

Proposition

(George and Veeramani [1]). A sequence $(x_n)_n$ *in* X *converges to* x *if and only if* $\lim_n M(x_n, x, t) = 1$*, for all* $t > 0$*.*

Definitions (George and Veeramani [1].)

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be M-Cauchy, or simply Cauchy, if for each $\epsilon \in]0, 1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$, or equivalently, if $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$.

X is said to be complete if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is also said to be complete.

Definition (Gregori and Romaguera [8].)

Let $(X, M, *)$ and $(Y, N, \diamond)$ be two fuzzy metric spaces. A mapping f from X to Y is said to be an isometry if for each $x, y \in X$ and $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$ and, in this case, if f is a bijection, X and Y are called isometric.

A fuzzy metric completion of (X, M) is a complete fuzzy metric space (X^*, M^*) such that X is isometric to a dense subspace of X^*.

X is said to be completable if it admits a fuzzy metric completion.

Proposition (Gregori and Romaguera [8].)

If a fuzzy metric space has a fuzzy metric completion then it is unique up to isometry.

Theorem (Gregori and Romaguera [9].)

A fuzzy metric space $(X, M, *)$ is completable if and only if it satisfies the following conditions:

(C1) Given two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X, then $\lim_n M(a_n, b_n, s) = 1$ for some $s > 0$ implies $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$.

(C2) Given two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X, then the assignment

$$t \rightarrow \lim_n M(a_n, b_n, t)$$

is a continuous function on $]0, \infty[$ with values in $]0, 1]$.

Example (Gregori and Romaguera [9].)

Let $\{x_n\}$ and $\{y_n\}$ be two strictly increasing sequences of positive real numbers, which converge to 1 with respect to the usual topology of $\mathbb{R}$, with $A \cap B = \emptyset$, where $A = \{x_n : n \in \mathbb{N}\}$ and $B = \{y_n : n \in \mathbb{N}\}$. Put $X = A \cup B$ and define a fuzzy set M on $X \times X \times]0, \infty[$ by:

$$M(x_n, x_n, t) = M(y_n, y_n, t) = 1 \text{ for all } n \in \mathbb{N}, t > 0,$$
$$M(x_n, x_m, t) = x_n \wedge x_m \text{ for all } n, m \in \mathbb{N} \text{ with } n \neq m, t > 0,$$
$$M(y_n, y_m, t) = y_n \wedge y_m \text{ for all } n, m \in \mathbb{N} \text{ with } n \neq m, t > 0,$$
$$M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m \text{ for all } n, m \in \mathbb{N}, t \geq 1,$$
$$M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m \wedge t \text{ for all } n, m \in \mathbb{N}, t \in]0, 1[.$$

$(X, M, *)$ is a fuzzy metric space, where $*$ is the minimum t-norm.
The authors observed that M satisfies condition $(C2)$, but it does not satisfy condition $(C1)$. Indeed, in [9] was observed that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X such that $\lim_n M(x_n, y_n, t) = 1$ for all $t \geq 1$, but $\lim_n M(x_n, y_n, t) = t$ for all $t \in]0, 1[$.

Example (Gregori and Romaguera [8].)

Let $\{x_n\}$ and $\{y_n\}$ be two sequences of distinct points such that $A \cap B = \emptyset$, where $A = \{x_n : n \geq 3\}$ and $B = \{y_n : n \geq 3\}$. Put $X = A \cup B$ and define a fuzzy set M on $X \times X \times]0, \infty[$ by:

$$M(x_n, x_m, t) = M(y_n, y_m, t) = 1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right],$$

$$M(x_n, y_m, t) = M(y_m, x_n, t) = \frac{1}{n} + \frac{1}{m},$$

for all $n, m \geq 3$.

$(X, M, *)$ is a fuzzy metric space, where $*$ is the Luckasievicz t-norm $(a * b = \max\{0, a + b - 1\})$.

M is stationary, and so it satisfies condition $(C1)$. However, M does not satisfy condition $(C2)$, since $\{x_n\}_{n \geq 3}$ and $\{y_n\}_{n \geq 3}$ are Cauchy sequences. Then

$$\lim_n M(x_n, y_n, t) = \lim_n \left(\frac{1}{n} + \frac{1}{n}\right) = 0.$$

Problem

To find a fuzzy metric space $(X, M, *)$ where for two M-Cauchy sequences $(a_n)_n$ and $(b_n)_n$ in X the assignment

$$t \mapsto \lim_n M(a_n, b_n, t) \text{ for all } t > 0,$$

does not define a continuous function on t, for the usual topology of $\mathbb{R}$.

Example (Gregori et al. [6].)

Let d be the usual metric on $\mathbb{R}$ restricted to $]0,1]$ and consider the standard fuzzy metric M_d induced by d. Put $X =]0,1]$ and define a fuzzy set M on $X \times X] \times]0,\infty[$ by

$$M(x,y,t) = \begin{cases} M_d(x,y,t), & 0 < t \leq d(x,y) \\ M_d(x,y,2t) \cdot \frac{t-d(x,y)}{1-d(x,y)} + M_d(x,y,t) \cdot \frac{1-t}{1-d(x,y)}, & d(x,y) < t \leq 1 \\ M_d(x,y,2t), & t > 1 \end{cases}$$

$(X, M, \cdot)$ is a fuzzy metric space.
Let $\{a_n\}$ and $\{b_n\}$, given by $a_n = \frac{1}{n}$ and $b_n = 1$ for all $n \in \mathbb{N}$. These are Cauchy sequences in $(X, M, \cdot)$ and the assignment

$$\lim_n M(a_n, b_n, t) = \begin{cases} \frac{t}{t+1}, & 0 < t < 1 \\ \frac{2t}{2t+1}, & t \geq 1 \end{cases}$$

is a well-defined function on $]0,\infty[$ which is not continuous at $t = 1$.

Theorem

A fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following three conditions are fulfilled:

- (c1) $\lim_n M(a_n, b_n, s) = 1$ for some $s > 0$ implies $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$.
- (c2) $\lim_n M(a_n, b_n, t) > 0$ for all $t > 0$.
- (c3) The assignment $t \to \lim_n M(a_n, b_n, t)$ for each $t > 0$ is a continuous function on $]0, \infty[$, provided with the usual topology of $\mathbb{R}$.

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