On a new attempt for generalizing fuzzy metric spaces

F.Castro, P. Tirado

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Y. Shen, H. Li, F. Wang, On Interval-Valued Fuzzy Metric Spaces, International Journal of Fuzzy Systems Vol.14,1 (2012), 35-44.

A new generalization of fuzzy metric spaces

A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994), 395-399.

- Interval-valued t-norm in interval numbers
- Follow step by step A. George and P. Veeramani

George and Veeramani's results [On some results in fuzzy metric spaces]:

- They define a (Hausdorff-)topology on a fuzzy metric space
- They show that every metric induces a fuzzy metric
- They prove Baire's theorem for fuzzy metric spaces

Shen, Li, and Wang's results [On Interval-Valued Fuzzy Metric Spaces]:

- They define a (Hausdorff-)topology on an Interval-Valued fuzzy metric space
- They show that every metric induces an Interval-Valued fuzzy metric
- They prove Baire's theorem for Interval-Valued fuzzy metric spaces

J. H. Park, Intuitionistic fuzzy metric spaces, Chaos, Solitons and Fractals 22 (2004) 1039-1046.

J.H. Park's results [Intuitionistic fuzzy metric spaces]:

- He define a (Hausdorff-)topology on an Intuitionistic fuzzy metric space
- He show that every metric induces an Intuitionistic fuzzy metric
- He prove Baire's theorem for Intuitionistic fuzzy metric spaces

V. Gregori, S. Romaguera and P. Veeramani, A note on intuitionistic fuzzy metric spaces, Chaos, Solitos and Fractals 28 (2006) 902-905.

Park's results follow directly from well-known theorems in fuzzy metric spaces

On a new attempt for generalizing fuzzy metric spaces

- Each interval-valued fuzzy metric space induces in a natural way two fuzzy metric spaces
- The topology generated by the interval-valued fuzzy metric coincides with the topology generated by one of its associated fuzzy metric
- Shen, Li and Wang's results follow directly from well-known results in fuzzy metric spaces

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Continuous t-norm

A t-norm is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions:

(i) $*$ is associative and commutative;

(ii) $a * 1 = a$ for every $a \in [0,1]$;

(iii) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$, for $a, b, c, d \in [0,1]$.

If, in addition, $*$ is continuous, then $*$ is called a continuous t-norm.

E.P. Klemant, R. Mesiar and E. Pap, Triangular Norms, Kluwer Academic Publishers, 2000.

Proposition

A t-norm is continuous if and only if it is continuous in each component

Paradigmatic examples of continuous t-norms are the minimum, denoted by $\wedge$, the usual product, denoted by $\cdot$ and the Lukasiewicz t-norm, denoted by $*_L$, where $x *_L y = \max\{x + y - 1, 0\}$. They satisfy the following well-known inequalities:

$x *_L y \leqslant x \cdot y \leqslant x \wedge y.$

In fact,

$x * y \leqslant x \wedge y$

for each t-norm $*$.

O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kibernetica 11 (1975) 326-334.

KM-Fuzzy metric space

A KM-fuzzy metric on a (non-empty) set X is a pair $(M,*)$ such that $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0,\infty)$ such that for all $x,y,z \in X$:

(KM1) $M(x,y,0) = 0$;

(KM2) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$;

(KM3) $M(x,y,t) = M(y,x,t)$ for all $t > 0$;

(KM4) $M(x,z,t+s) \geqslant M(x,y,t) * M(y,z,s)$ for all $t,s > 0$;

(KM5) $M(x,y,\_)$ is left continuous on $(0,\infty)$.

A triple $(X,M,*)$ where X is a (non-empty) set and $(M,*)$ is a KM-fuzzy metric on X, is said to be a KM-fuzzy metric space.

A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994), 395-399.

GV-Fuzzy metric space

A GV-fuzzy metric on a (non-empty) set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ such that for all $x, y, z \in X; t, s > 0$:

(GV1) $M(x, y, t) > 0$;

(GV2) $M(x, y, t) = 1$ if and only if $x = y$;

(GV3) $M(x, y, t) = M(y, x, t)$;

(GV4) $M(x, z, t + s) \geqslant M(x, y, t) * M(y, z, s)$;

(GV5) $M(x, y, \_)$ is continuous on $(0, \infty)$.

A triple $(X, M, *)$ where X is a (non-empty) set and $(M, *)$ is a GV-fuzzy metric on X, is said to be a GV-fuzzy metric space.

Obviously, each GV-fuzzy metric can be considered as a KM-fuzzy metric by defining $M(x, y, 0) = 0$ for all $x, y \in X$.

Every fuzzy metric $(M, *)$ on X generates a Hausdorff first countable topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, r, t) : x \in X,\ r \in (0, 1),\ t > 0\}$, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for all $x \in X$, $r \in (0, 1)$and $t > 0$.

V.Gregori, S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets and Systems, vol. 115, pp. 485-489, 2000.

Theorem

Let $(X, M, *)$ be a fuzzy metric space. Then (X, τ_M) is a metrizable topological space.

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$$I = [0, 1]$$

$$[I] = \{\overline{a} = [a^-, a^+] : 0 \leq a^- \leq a^+ \leq 1\}.$$

If $a^- = a^+$, then the interval number $\overline{a}$ degenerates into an ordinary real number on I.

Conversely, every $a \in I$ induces the interval number $[a, a]$ that we will denote as $\overline{a}$ if no confusion arises.

We will write $(I] = [I] - \{\overline{0}\}$ and $(I) = [I] - \{\overline{0}, \overline{1}\}$.

(i) $\overline{a} \wedge \overline{b} = [a^- \wedge b^-, a^+ \wedge b^+]$;

(ii) $\overline{a} \vee \overline{b} = [a^- \vee b^-, a^+ \vee b^+]$;

(iii) $\overline{a}^c = \overline{1} - \overline{a} = [1 - a^+, 1 - a^-]$.

Y. Shen, H. Li, F. Wang, On Interval-Valued Fuzzy Metric Spaces, International Journal of Fuzzy Systems Vol.14,1 (2012), 35-44.

Definition

An $\mathscr{I}\mathscr{V}$-t-norm is a binary operation $\overline{*}: [I] \times [I] \to [I]$ that satisfies the following conditions (i) $\overline{*}$ is associative and commutative; (ii) $\overline{a} \overline{*} \overline{1} = \overline{a}$ and $\overline{a} \overline{*} I = [0, a^+]$ for every $\overline{a} = [a^-, a^+] \in [I]$; (iii) $\overline{a} \overline{*} \overline{b} \leq \overline{c} \overline{*} \overline{d}$ whenever $\overline{a} \leq \overline{c}$ and $\overline{b} \leq \overline{d}$, for $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in [I]$. If, in addition, $\overline{*}$ is continuous in its first component, then $\overline{*}$ is called a continuous $\mathscr{I}\mathscr{V}$-t-norm.

Proposition

An $\mathscr{I}\mathscr{V}$-t-norm $\overline{*}$ is continuous if and only if it is continuous in its first component.

Proposition

Every $\mathscr{I}\mathscr{V}-$t-norm $\overline{*}$ acts componentwise.

Given an $\mathscr{I}\mathscr{V}$-t-norm $\overline{*}$ we can write $\overline{*} = [*^-, *^+]$ were $*^-$ and $*^+$ are two continuous t-norms such that $*^- \leq *^+$. In fact $\overline{\wedge} = [\wedge, \wedge]$ and $\bar{\cdot} = [\cdot, \cdot]$.

Y. Shen, H. Li, F. Wang, On Interval-Valued Fuzzy Metric Spaces, International Journal of Fuzzy Systems Vol.14,1 (2012), 35-44.

Definition

An $\mathscr{I}\mathscr{V}$-fuzzy metric space is a triple $(X, \overline{M}, \overline{*})$ such that X is a non-empty set, $\overline{*}$ is a continuous $\mathscr{I}\mathscr{V}$ t-norm and $\overline{M}$ is an interval-valued fuzzy set on $X \times X \times (0, \infty)$ such that for all $x, y, z \in X$; $t, s > 0$:

(a) $\overline{M}(x, y, t) > \overline{0}$;

(b) $\overline{M}(x, y, t) = \overline{1}$ if and only if $x = y$;

(c) $\overline{M}(x, y, t) = \overline{M}(y, x, t)$;

(d) $\overline{M}(x, z, t + s) \geq \overline{M}(x, y, t) \overline{*} \overline{M}(y, z, s)$;

(e) $\overline{M}(x, y, \_) : (0, \infty) \to (I]$ is continuous;

$\overline{M} = [M^{-}, M^{+}]$ is called an interval-valued fuzzy metric on X

Conditions in Definition of $\mathscr{I}\mathscr{V}$-fuzzy metric space together with previous Proposition , where $\overline{*} = [*^-, *^+]$, imply that $(X, M^-, *^-)$ and $(X, M^+, *^+)$ are fuzzy metric spaces.

Each $\mathscr{I}\mathscr{V}$-fuzzy metric $\overline{M}$ on X generates a Hausdorff first countable topology $\tau_{\overline{M}}$ on X which has as a base the family of open sets of the form $\{B_{\overline{M}}(x, \overline{r}, t) : x \in X, \overline{0} < \overline{r} < \overline{1}, t > 0\}$, where $B_{\overline{M}}(x, \overline{r}, t) = \{y \in X : \overline{M}(x, y, t) > \overline{1} - \overline{r}\}$ for all $x \in X$, $\overline{0} < \overline{r} < \overline{1}$ and $t > 0$.

Proposition

Let $(X, \overline{M}, \overline{*}) = (X, [M^-, M^+], [*^-, *^+])$ *be an* $\mathscr{I}\mathscr{V}$*-fuzzy metric space. Then, for each* $x \in X$, $r \in (0,1)$, $t > 0$ *we have* $B_{\overline{M}}(x, \overline{r}, t) = B_{M^-}(x, r, t)$.

Theorem

Let $(X, \overline{M}, \overline{*}) = (X, [M^-, M^+], [*^-, *^+])$ *be an* $\mathscr{I}\mathscr{V}$*-fuzzy metric space. Then the topologies* $\tau_{\overline{M}}$ *and* τ_{M^-} *coincide on* X.

Corollary

Let $(X, \overline{M}, \overline{*}) = (X, [M^-, M^+], [*^-, *^+])$ *be an* $\mathscr{I}\mathscr{V}$*-fuzzy metric space. Then* $(X, \tau_{\overline{M}})$ *is a metrizable topological space.*

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