Non-standard applications of fractal dimension through fractal structures

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International Summer Workshop in Applied Topology (ISWAT'14)

Outline

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Fractal structures

Fractal structures were introduced to characterize non-Archimedean quasimetrization, though they can be used in different topics.

Some of them are:

1. Non-Archimedean quasi-metrization.
2. Metrization.
3. Topological dimension.
4. Self-similar sets (fractals).
5. **Fractal dimension.**
6. Peano continua.
7. Compactification.
8. Completeness.
9. Transitive quasi-uniformities.
10. Inverse limits.

Definitions

Notations

Let Γ be a covering on a topological space X.

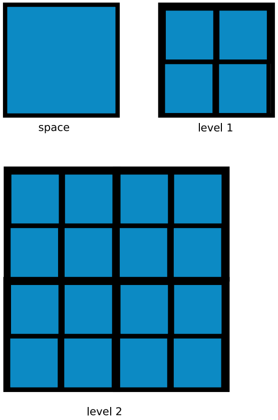
- $U_\Gamma = \{(x,y) \in X \times X : y \notin \bigcup\{A \in \Gamma : x \notin A\}\}$.
- $\Gamma_1 \prec\prec \Gamma_2 \iff \Gamma_1 \prec \Gamma_2$ and for each $B \in \Gamma_2$ it holds $B = \bigcup\{A \in \Gamma_1 : A \subseteq B\}$.

Definitions

Definition

A **fractal structure** is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that $\Gamma_{n+1} \prec\prec \Gamma_n$.
A fractal structure induces a transitive base of quasi-uniformity, given by $\{U_{\Gamma_n} : n \in \mathbb{N}\}$.

Example of a fractal structure



The natural fractal structure on Euclidean spaces

Definition

The **natural fractal structure** on an Euclidean space $\mathbb{R}^d$ is defined as the family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, where its levels are given by $\Gamma_n = \left\{ \left[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}\right] \times \left[\frac{k_2}{2^n}, \frac{k_2+1}{2^n}\right] \times \ldots \times \left[\frac{k_d}{2^n}, \frac{k_d+1}{2^n}\right] : k_i \in \mathbb{Z}, i \in \{1, \ldots, d\} \right\}$ for all $n \in \mathbb{N}$.

Outline

1. Fractal structures
2. Fractal dimensions I and II
   - Box-counting dimension
   - Fractal dimensions I & II
   - Application to the domain of words
3. Fractal dimensions III and IV
   - Hausdorff dimension
   - Fractal dimension III
   - The fractal dimension of a curve
   - Studying the Hurst exponent of random processes
   - Fractal dimension IV

Box-counting dimension

Definition

The **box-counting dimension** of $F \subseteq \mathbb{R}^d$ is defined by

$$\dim_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

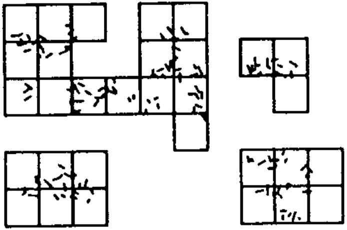


Figure: $N_\delta(F)$ is the number of δ-cubes that meet F.

Fractal dimensions I & II

Definition

1. Let (X, Γ) be a GF-space. The **fractal dimension I** of $F \subseteq X$ is defined as
$$\dim_{\Gamma}^{1}(F) = \lim_{n\to\infty} \frac{\log N_n(F)}{n \log 2}.$$
2. Let Γ be a fractal structure on a metric space (X, ρ). The **fractal dimension II** of $F \subseteq X$ is given by
$$\dim_{\Gamma}^{2}(F) = \lim_{n\to\infty} \frac{\log N_n(F)}{-\log \delta(F, \Gamma_n)}.$$

$N_n(F)$ is the number of elements of Γ_n which meet F.
$\delta(F, \Gamma_n) = \sup\{\mathrm{diam}\,(A) : A \in \Gamma_n \text{ and } A \cap F \neq \emptyset\}$.

The domain of words

- The domain of words appears when modeling the streams of information in Kahn's model of parallel computation.
- Let Σ be a finite non-empty alphabet (set) and let $\Sigma^\infty = \cup_{n\in\mathbb{N}}\Sigma^n \cup \Sigma^{\mathbb{N}}$ be the collection of all finite and infinite sequences (called *words*) over Σ. Let ε be the empty word.
- The prefix order $\sqsubseteq$ is defined on Σ^∞ as usual: $x \sqsubseteq y$ iff x is a prefix of y. We also denote by $x \sqcap y$ to the common prefix of x and y.
- For each $x \in \Sigma^\infty$, let $l(x)$ be its length, where $l(\varepsilon) = 0$.

Quasi-metrics & fractal structures on the domains of words

- Let us define a (non-Archimedean) quasi-metric d on Σ^∞ as follows:

$$d(x,y) = \begin{cases} 0 & \ldots \quad x \sqsubseteq y \\ 2^{-l(x \sqcap y)} & \ldots \quad \text{otherwise} \end{cases}$$

- Any non-Archimedean quasi-metric d induces a fractal structure $\mathbf{\Gamma}_d = \{\Gamma_n : n \in \mathbb{N}\}$, whose levels are given as $\Gamma_n = \{B_{d^{-1}}(x, \frac{1}{2^n}) : x \in X\}$.
- In this case, we can describe the induced fractal structure as follows: $\Gamma_n = \{w^\# : w \in \Sigma^n\} \cup \{w^\sqsubseteq : w \in \Sigma^k : k < n\}$, where
$w^\sqsubseteq = \{u \in \Sigma^k : k \leq n, u \sqsubseteq w\}$, and
$w^\# = \{wu : u \in \Sigma^\infty\} \cup w^\sqsubseteq$.

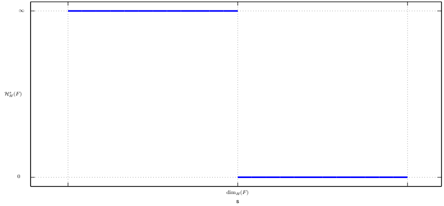
Application to search trees

1. Study of the search tree of possible configurations in *Othello's* board game.
2. Define a subset of the domain of words induced by a tree or a node of the tree.
3. Calculate the fractal dimension of the subset.
4. The Othello's search tree has a strong fractal pattern.

Outline

Hausdorff dimension (1919)

- A countable family of subsets $\{U_i\}_{i\in I}$ is a $\delta-$cover of $F \subseteq X$, if it verifies that $F \subseteq \bigcup_{i\in I} U_i$ with $\mathrm{diam}\,(U_i) \leq \delta$.
- $\mathcal{H}^s_\delta(F) = \inf\left\{ \sum_{i=1}^{\infty} \mathrm{diam}\,(U_i)^s : \{U_i\}_{i\in I} \text{ is a } \delta- \text{ cover of } F\right\}$.
- $s-$dimensional **Hausdorff measure** of F: $\mathcal{H}^s_H(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F)$.
- $\dim_H(F) = \inf\{s : \mathcal{H}^s_H(F) = 0\} = \sup\{s : \mathcal{H}^s_H(F) = \infty\}$.



Fractal dimension III

Definition

Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), and take $F \subseteq X$ such that $\delta(F,\Gamma_n) \to 0$. Consider $\mathcal{H}^s_{n,3}(F) = \sum\{\mathrm{diam}\,(A)^s : A \in \Gamma_n \text{ and } A \cap F \neq \emptyset\}$ and define $\mathcal{H}^s_3(F) = \lim_{n\to\infty} \mathcal{H}^s_{n,3}(F)$.

$$\dim^3_{\mathbf{\Gamma}}(F) = \inf\{s : \mathcal{H}^s_3(F) = 0\} = \sup\{s : \mathcal{H}^s_3(F) = \infty\}$$

Fractal dimension III version for curves

Definition

Let $\alpha : [0,1] \to \mathbb{R}^n$ be a parametrization of a curve and let $\mathbf{\Gamma}$ be the natural fractal structure on $[0,1]$. Then the fractal structure induced by $\mathbf{\Gamma}$ on the image set $\alpha([0,1]) \subseteq \mathbb{R}^n$ is defined as the countable family of coverings $\mathbf{\Delta} = \{\Delta_n : n \in \mathbb{N}\}$, whose levels are given by

$$\Delta_n = \alpha(\Gamma_n) = \{\alpha(A) : A \in \Gamma_n\}.$$

Fractal dimension III version for curves

Definition

Let $\alpha : [0,1] \to \mathbb{R}^n$ be a parametrization of a real curve and let $\mathbf{\Gamma}$ be the natural fractal structure on $[0,1]$. Let also $\mathbf{\Delta}$ be the fractal structure induced by $\mathbf{\Gamma}$ on the image set $\alpha([0,1]) \subseteq \mathbb{R}^n$.
We define the fractal dimension of α with respect to $\mathbf{\Delta}$ by

$$\dim_{\mathbf{\Delta}}(\alpha) = \dim_{\mathbf{\Delta}}^3(\alpha([0,1])).$$

Properties of the fractal dimension for curves

Proposition

Let $\alpha : [0,1] \to \mathbb{R}$ *be a parametrization of a real curve equipped with its natural fractal structure* $\mathbf{\Gamma}$. *Let also* $\mathbf{\Delta}$ *be the fractal structure induced by* $\mathbf{\Gamma}$ *on* $\alpha([0,1]) \subseteq \mathbb{R}$. *Then*

- *If* α *is constant, then* $\dim_{\mathbf{\Delta}}(\alpha) = 0$.
- *If* α *is continuous and not constant, then* $\dim_{\mathbf{\Delta}}(\alpha) \geq 1$.
- *If* α *is differentiable and not constant, then* $\dim_{\mathbf{\Delta}}(\alpha) = 1$.

Random process

Let $(\mathbf{X}, \mathcal{A}, P)$ be a probability space and let $t \in [0, \infty)$ denote time. We say that $\mathbf{X} = \{X(t, \omega) : t \geq 0\}$ is a **random process** or a random function from $[0, \infty) \times \Omega$ to $\mathbb{R}$, if $X(t, \omega)$ is a random variable for all $t \geq 0$ and all $\omega \in \Omega$ (ω belongs to a sample space Ω). We think of $\mathbf{X}$ as defining a sample function $t \mapsto X(t, \omega)$ for all $\omega \in \Omega$. Thus, the points of Ω parametrize the functions $\mathbf{X} : [0, \infty) \times \Omega \to \mathbb{R}$ and P is a probability measure on this class of functions.

Random process

The increments of a random function $X(t,\omega)$ are said to be:

- *stationary*, if for each $a > 0$ and $t \geq 0$

$$X(a+t,\omega) - X(a,\omega) \sim X(t,\omega) - X(0,\omega).$$

- *self-affine with parameter* $H \geq 0$, if for any $h > 0$ and any $t_0 \geq 0$,

$$X(t_0+\tau,\omega) - X(t_0,\omega) \sim \frac{1}{h^H}\Big\{X(t_0+h\tau,\omega) - X(t_0,\omega)\Big\}.$$

Fractal dimension III for curves vs. Hurst exponent

Theorem

Let $\alpha : [0,1] \to \mathbb{R}$ *be a sample function of a random process* $\mathbf{X}$ *with stationary and self-affine increments with parameter* H*. Let* $\mathbf{\Gamma}$ *be the natural fractal structure on* $[0,1]$*. Then* $\dim_{\mathbf{\Gamma}}(\alpha) = 1/H$*.*

Fractal dimension III for curves vs. Hurst exponent

Theorem

Let $\alpha : [0,1] \to \mathbb{R}$ be a sample function of a random process $\mathbf{X}$ *and let* $\mathbf{\Gamma}$ *be the natural fractal structure on* $[0,1]$*. Suppose that there exists* $s > 0$ *such that*

1. *there exists* $m_s(M(\frac{1}{2^n}, \omega))$*, and*
2. $m_s(M(\frac{1}{2^n}, \omega)) = 2\ m_s(M(\frac{1}{2^{n+1}}, \omega))$

for all $n \in \mathbb{N}$*. Then* $\dim_{\mathbf{\Gamma}}(\alpha) = s$*.*

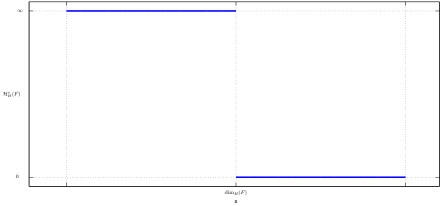
Given a random function $X(t, \omega)$, its **cumulative range** is

$$M(t, T, \omega) = \sup_{s \in [t, t+T]} \Big\{ Y(s, t, \omega) \Big\} - \inf_{s \in [t, t+T]} \Big\{ Y(s, t, \omega) \Big\},$$

where $Y(s, t, \omega) = X(s, \omega) - X(t, \omega)$ and $M(T, \omega) = M(0, T, \omega)$.

Hausdorff dimension

- A countable family of subsets $\{U_i\}_{i\in I}$ is a $\delta-$cover of $F \subseteq X$, if it verifies that $F \subseteq \bigcup_{i\in I} U_i$ with $\mathrm{diam}\,(U_i) \leq \delta$.
- $\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} \mathrm{diam}\,(U_i)^s : \{U_i\}_{i\in I} \text{ is a } \delta- \text{ cover of } F \right\}$.
- $s-$dimensional **Hausdorff measure** of F:
  $\mathcal{H}^s_H(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F)$.
- $\dim_H(F) = \inf\{s : \mathcal{H}^s_H(F) = 0\} = \sup\{s : \mathcal{H}^s_H(F) = \infty\}$.



Fractal dimension IV

Definition

$$\mathcal{H}_n^s(F) = \inf\left\{\sum_{i\in I} \mathrm{diam}\,(A_i)^s : A_i \in \bigcup_{l\geq n}\Gamma_l, F \subseteq \bigcup_{i\in I} A_i, |I| < \infty\right\}.$$

$$\mathcal{H}^s(F) = \lim_{n\to\infty} \mathcal{H}_n^s(F)$$

The **fractal dimension IV** of F is defined by

$$\dim_{\mathbf{\Gamma}}(F) = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

Connection between fractal dimension IV and Hausdorff dimension

Theorem

Let $\mathbf{\Gamma}$ *be the natural fractal structure on the Euclidean space* $\mathbb{R}^d$ *and let* F *be a bounded subset of* $\mathbb{R}^d$. *Then*

$$\dim_{\mathbf{\Gamma}}(F) = \dim_H(\overline{F}).$$

Corollary

Let $\mathbf{\Gamma}$ *be the natural fractal structure on the Euclidean space* $\mathbb{R}^d$ *and let* F *be a* **compact** *subset of* $\mathbb{R}^d$. *Then*

$$\dim_{\mathbf{\Gamma}}(F) = \dim_H(F).$$

Related works (I)

- **Fractal structures:** F.G. Arenas and M.A. Sánchez-Granero, *A characterization of non-archimedeanly quasimetrizable spaces,* Rend. Istit. Mat. Univ. Trieste XXX (1999) 21-30.
- **Fractal dimensions I & II:** M. Fernández-Martínez and M.A. Sánchez-Granero, *Fractal dimension for fractal structures*, Topology Appl. 163 (2014) 93-111.
- **Fractal dimension III:** M. Fernández-Martínez and M.A. Sánchez-Granero, *Fractal dimension for fractal structures: A Hausdorff approach,* Topology Appl. 159 (2012) 1825-1837.
- **Fractal dimension IV:** M. Fernández-Martínez and M.A. Sánchez-Granero, *Fractal dimension for fractal structures: A Hausdorff approach revisited,* J. Math. Anal. Appl. 409 (2014) 321-330.

Related works (II)

- **Domain of words:** M. Fernández-Martínez, M.A. Sánchez-Granero and J.E. Trinidad Segovia, *Fractal dimension for fractal structures: applications to the domain of words*, Appl. Math. Comput. 219 (2012) 1193-1199.
- **FD algorithms:** M.A. Sánchez-Granero, M. Fernández-Martínez and J.E. Trinidad Segovia, *Introducing fractal dimension algorithms to calculate the Hurst exponent of financial time series,* Eur. Phys. J. B (2012) 85: 86.
- **Calculating the Hausdorff dimension:** M. Fernández-Martínez, M.A. Sánchez-Granero, *How to calculate the Hausdorff dimension using fractal structures*, preprint.
- M. Fernández-Martínez, M.A. Sánchez-Granero, J.E. Trinidad Segovia, *Fractal dimensions for fractal structures and their applications to financial markets,* Aracne editrice S.r.l, Roma, 2013.