Non differentiability of Pettis primitives

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Preliminaries

Throughout X denotes a Banach space unless otherwise indicated. $\| \ \|$ denotes the norm of this Banach space.

We consider functions from $[0,1]$ into X. We use the standard Lebesgue measure λ on $[0,1]$.

Let $f : [0,1] \to X$. Function f is strongly measurable if there exists a sequence of simple functions $\{f_n\}$ converging a.e. to f. Moreover, if $\{f_n\}$ satisfies the additional property that

$$lim_{n\to\infty} \int_{[0,1]} \|f - f_n\| d\lambda = 0,$$

then we say that f is Bochner integrable. In such a case, we let

$$(B)\int_{[0,1]} f = \lim_{n\to\infty} \int_{[0,1]} f_n.$$

Recall that $(B)\int f$ is independent of the choice of $\{f_n\}$.

As usual, we let X^* denote the dual of X.

We say that $f : [0,1] \to X$ is weakly measurable, if x^*f is λ-measurable for all $x^* \in X^*$.

We call a weakly measurable function $f : [0,1] \to X$ Pettis integrable if for each measurable set $A \subseteq [0,1]$ there is $x_A \in X$ such that

$$x^*(x_A) = \int_A x^* f d\lambda,$$

for all $x^* \in X^*$.

In the case that f is Pettis integrable, we use

$$(P)\int_{[0,1]} f$$

to denote the Pettis integral of f.

Basics Properties of Bochner Integral

and

Pettis Integrals

We next recall some basic well-known facts and differences between the Bochner integral and Pettis integral.

Fact 1 *For any X,*

f is Bochner integrable $\Rightarrow f$ is Pettis integrable.

Fact 2 *For X finite dimensional (i.e., $X = \mathbb{R}^n$) we have the following:*

$$f \text{ is Lebesgue}$$
$$\Updownarrow$$
$$f \text{ is Bochner integrable}$$
$$\Updownarrow$$
$$f \text{ is Pettis integrable}$$

Moreover, values of $\int f$ under all of the above integrals coincide.

Fact 3 *There are weakly measurable functions which are not strong measurable.*

Fact 4 *The following are equivalent:*

- f *is strongly mesurable.*
- f *is weakly measurable and the range of* f *is essentially separable, i.e., there exists a set* E *of measure zero such that* $f([0,1] \setminus E)$ *is separable.*

Corollary 5 *If* X *is a separable Banach space, then strong measurability and weak measurability coincide.*

Fact 6 *Let* $f : [0,1] \to X$ *be strongly measurable. Then,*

$$f \text{ is Bochner integrable} \Updownarrow \int \|f\| < \infty$$

Fact 7 *Let* $f : [0,1] \to X$ *be Bochner and* F *be its primitive, i.e.,* F *is the* X*-valued measure on* $[0,1]$ *defined by*

$$(B)\int_A f = F(A).$$

Then, F *is countably additive,* λ*-continuous and is of finite variation.*

Moreover, for almost every $t \in [0,1]$*, we have that*

$$\lim_{h\to 0} \frac{F(t+h) - F(t)}{h} = f(t).$$

Fact 8 *Let* $f : [0, 1] \to X$ *be Pettis and* F *be its primitive, i.e.,* F *is the* X*-valued measure on* $[0, 1]$ *defined by*

$$(P) \int_A f = F(A).$$

Then, F *is countably additive,* λ*-continuous and is of* σ*-finite variation.*

Theorem 9 *(Dilworth-Girardi, 1995)*
Let X *be any infinite dimensional Banach space. Then, there exists Pettis integrable* $f : [0, 1] \to X$ *such that* F*, the primitive of* f*, is nowhere differentiable! More precisely, for every* $t \in [0, 1]$ *we have that*

$$\lim_{h \to 0} \left\| \frac{F(t + h) - F(t)}{h} \right\| = \infty.$$

Motivation

It is generally considered that the class $\mathcal{P}$ of Pettis integrable functions is much larger than the class $\mathcal{B}$ of Bochner integrable functions whenever X is infinite dimensional.

Indeed, if X is not separable then, the class of strongly measurable functions is much smaller than the class of weakly measurable functions as the range of strongly measurable functions is essentially separable.

However, even when X is separable and infinite dimensional class $\mathcal{P}$ is considered much smaller than the class $\mathcal{B}$.

How does one make this statement precise?

A natural notion of smallness in a complete metric space is that of first category.

The space $\mathcal{B}$ is a Banach space, however, the class $\mathcal{P}$ enjoys no natural complete metric. Hence, it is difficult to make a meaningful use of category in this setting.

Moreover, given any Bochner primitive $F \in C([0,1],X)$, arbitrarily close to it, in the norm of $C([0,1],X)$, there are Pettis primitives.

However, both of $\mathcal{B}$ and $\mathcal{P}$ are vector spaces. Our investigation was motivated by the following question:

How large a set $\mathcal{A} \subset \mathcal{P}$ must be so that $< \mathcal{B} \cup \mathcal{A} >= \mathcal{P}$?

A large subset of $\mathcal{P} \setminus \mathcal{B}$

The concept of unconditional convergence is central to the investigation of Pettis integral.

Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is unconditionally convergent if every rearrangement of $\{x_n\}$ converges.

If X is finite dimensional, then $\{x_n\}$ is unconditionally convergent iff $\{x_n\}$ is absolutely convergent.

Theorem 10 *(Dvoretzky-Rogers) If X is infinite dimensional, then there is a sequence $\{x_n\}$ in X which converges unconditionally but does not converge absolutely.*

Fact 11 *Let* $\{E_n\}$ *be a partition of* $[0,1]$ *and let* $\{x_n\}$ *be a sequence in* X*.*

- $\sum_{n=1}^{\infty} x_n \chi_{E_n}$ *is Bochner integrable iff* $\sum_{n=1}^{\infty} \|x_n\| \lambda(E_n)$.

- $\sum_{n=1}^{\infty} x_n \chi_{E_n}$ *is Pettis integrable iff* $\sum_{n=1}^{\infty} x_n \lambda(E_n)$ *is unconditionally convergent.*

In particular, if $\sum_{n=1}^{\infty} x_n \lambda(E_n)$ *converges unconditionally but not absolutely, then* $\sum_{n=1}^{\infty} x_n \chi_{E_n}$ *is Pettis integrable but not Bochner integrable.*

Theorem 12 *(BDDi, Main theorem) There exists a subset $\mathcal{K} \subset \mathcal{P}$ of the cardinality continuum such that the following holds:*

- *Each $f \in \mathcal{K}$ is nowhere differentiable.*

- *If $f_1, \ldots, f_n \in \mathcal{K}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are not all zero, then*

$$\sum_{i=1}^{n} \lambda_i f_i \notin \mathcal{B}.$$

Theorem 13 *(BDDi, First Approximation) There exists a subset $\mathcal{K} \subset \mathcal{P}$ of the cardinality continuum such that the following holds:*

- *If $f_1, \ldots, f_n \in \mathcal{K}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are not all zero, then*

$$\sum_{i=1}^{n} \lambda_i f_i \notin \mathcal{B}.$$

Theorem 14 *(BDDi, Second Approximation) There exists a subset $\mathcal{K} \subset \mathcal{P}$ of the cardinality continuum such that the following holds:*

- *For each $f \in \mathcal{K}$ and subinterval I of $[0, 1]$ we have that $f_{|I}$ is not Bochner.*

- *If $f_1, \dots, f_n \in \mathcal{K}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ are not all zero, then*
$$\sum_{i=1}^{n} \lambda_i f_i \notin \mathcal{B}.$$

Proof of the First Approximation. Recall that there is a set a $\mathcal{A}$ of cardinality continuum such that each $A \in \mathcal{A}$ is an infinite subset of $\mathbb{N}$ and if A, B are distinct elements of $\mathcal{A}$, then $A \cap B$ is finite. (Such a collection is called an almost disjoint collection.)

Let $\{x_n\}$ be an unconditionally convergent sequence X which does not converge absolutely. Let $I_1, I_2, \ldots$ be a parition of $[0, 1)$ into nonoverlapping intervals. For each $A \in \mathcal{A}$, we define a function $f_A \in \mathcal{P} \setminus \mathcal{B}$ in the following fashion. List A as $a_1, a_2, \ldots$ in an increasing order.

Then,

$$f_A \equiv \sum_{i=1}^{\infty} x_i \frac{1}{|I_{a_i}|} \chi_{I_{a_i}}.$$

Let $\mathcal{K} = \{f_A : A \in \mathcal{A}\}$. Clearly, $f_A \in \mathcal{P} \setminus \mathcal{B}$. Let $f_{A_1}, \ldots f_{A_n}$ be distinct elements of $\mathcal{K}$ and $\lambda_1, \ldots \lambda_n \in \mathbb{R}$ be non zeros. As the collection $\mathcal{A}$ is almost disjoint, there is $n \in A_1$ such that for $(n, \infty) \cap A_1 \cap A_i = \emptyset$, for $2 \leq i \leq n$. Then the function $\sum_{i=1}^{n} \lambda_i f_i$ is equal to $\lambda_1 f_1$ on I_m for all $m \in A_1$, $m > n$. Hence, we have that $\sum_{i=1}^{n} \lambda_i f_i$ is not in $\mathcal{B}$. End of proof of the First Approximation.

Proof of the Second approximation. We proceed with some notations and then lemmas.

Let $n_1, n_2, \ldots$ be a sequence of integers greater than 1. Then, $(n_1, n_2, \ldots)$-*tree*, denoted by $\mathcal{T}(n_1, n_2, \ldots)$, is defined as

$$\mathcal{T}(n_1, n_2, \ldots) = \{\emptyset\} \bigcup_{i=1}^{\infty} \Pi_{j=1}^{i}\{0, \ldots, n_j - 1\}.$$

If $\sigma \in \mathcal{T}(n_1, n_2, \ldots)$, then $|\sigma|$ denotes the length of σ and if $n \leq |\sigma|$, then $\sigma_{|n}$ denotes the restriction of σ to the first n terms. If $\sigma_{|n} = \tau$, then we say that σ is an extension of τ or τ is a restriction of σ. If $\sigma \in \mathcal{T}(n_1, n_2, \ldots)$ and $i \in \mathbb{N}$, then σi is a extension of σ of length $n+1$ whose $n+1$ term is i. If $\sigma \in \mathcal{T}(n_1, n_2, \ldots)$, then

$[\sigma] = \{\tau \in \mathcal{T}(n_1, n_2, \ldots) : \tau$ is an extension of $\sigma.\}$

Associated with $(n_1, n_2, \ldots)$, we also define an interval function. More specifically, with each $\sigma \in \mathcal{T}(n_1, n_2, \ldots)$, we associate I_σ an subinterval of $[0, 1]$ such that the following holds.

1. For each k, $\{I_\sigma : \sigma \in \mathcal{T}(n_1, n_2, \ldots), |\sigma| = k\}$ is partition of $[0, 1]$ into $n_1 \cdot n_k$ non-overlapping closed intervals.

2. For each $\sigma \in \mathcal{T}(n_1, n_2, \ldots)$ with $|\sigma| = k$, we have that $\{I_{\sigma l} : 0 \leq l \leq n_{k+1} - 1\}$ is a partition of I_σ into non-overlapping intervals of equal length.

Lemma 15 *There exists a function* $T : \mathcal{T}(n_1, n_2, \ldots) \to X$ *such that the following condition holds.*

For all $\sigma \in \mathcal{T}(n_1, n_2, \ldots)$*, and any bijection* $\alpha : \mathbb{N} \to [\sigma]$*, the sequence* $\{x_n\}$ *defined by* $x_n = T(\alpha(n))$ *converges unconditionally but not absolutely.*

Moreover, we can make the function T *1-1, if necessary.*

Lemma 16 *Let* $\{A_\sigma\}_{\sigma \in \mathcal{T}(n_1, n_2, \ldots)}$ *be a pairwise disjoint system of positive measure subsets of* $[0, 1]$ *such that* $A_\sigma \subseteq I_\sigma$ *and let* T *be a function of the Lemma 15. Then,*

$$f = \sum_{\sigma \in \mathcal{T}(n_1, n_2, \ldots)} T(\sigma) \frac{1}{\lambda(A_\sigma)} \chi_{A_\sigma}$$

is Pettis but Bochner on no subinterval of $[0, 1]$*.*

The rest of the proof of the Second Approximation Theorem on the board.

Now, the proof of the Main Theorem follows from techniques of the above theorems and the following important theorem of Dvoretzky's Theorem:

Theorem 17 *(Dvoretzky's Theorem)*
Let $\epsilon > 0$ *and* $K > 0$. *Then, there is* $N(\epsilon, K) > 0$ *such that for any normed linear space* X *with dimension greater than* $N(\epsilon, K)$, *there is an isomorphism* $T : E_K \to X$ *such that* $1 \leq \|T\| < 1 + \epsilon$.