Multivalued F-Contractions and Some Fixed Point Results

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Introduction and preliminaries

Fixed point theory has various applications in many different fields of mathematics such as nonlinear functional analysis, mathematical analysis, operator theory and general topology. The fixed point theory is divided into three major areas:

- First is the topological fixed point theory, which attributed to the work of Brouwer in 1910, who proved that any continuous self-map of the closed unit ball of $\mathbb{R}^n$ has a fixed point. The results of Schauder (1930), Darbo (1955), Krasnoselskii (1955) and Mönch (1980) are related to these directions.
- Second is the discrete fixed point theory, which begins to the work of Kneser in 1950, who proved that: Let $(X, \preceq)$ be a partially ordered set and T be a self mapping of X such that $x \preceq Tx$ for all $x \in X$. If every chain in X has a supremum, then T has a fixed point. The results of Tarski (1955) and Aman (1977) are related to these directions.
- Third is the metrical fixed point theory on contraction or contraction type mappings on complete metric spaces. The metrical fixed point theory based on the Banach Contraction Principle, published in 1922.

Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be a contraction (ordinary contraction) mapping if there exists a constant $L \in [0,1)$, called a contraction factor, such that

$$d(Tx, Ty) \leq Ld(x,y) \text{ for all } x, y \in X. \quad (1)$$

Banach Contraction Principle says that any contraction self-mappings on a complete metric space has a unique fixed point. This principle is one of a very power test for existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, Banach Contraction Principle has been extended and generalized in many directions.

One of the most interesting generalization of it was given by Wardowski [1]. First we recall the concept of F-contraction, which was introduced by Wardowski [1], later we will mention his result.

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[1] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012, 2012:94, 6 pp.

- Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

$(F1)$ F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,
$(F2)$ For each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,
$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

- Some examples of the functions belonging $\mathcal{F}$ are $F_1(\alpha) = \ln \alpha$, $F_2(\alpha) = \alpha + \ln \alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$, $F_4(\alpha) = \ln\left(\alpha^2 + \alpha\right)$ and $F_5(\alpha) = -\frac{1}{\arctan \alpha^t}$ for $t \in (0, 1)$.

Definition (Wardowski [1])

Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be an F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)) \tag{2}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

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[1] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012, 2012:94, 6 pp.

- From $(F1)$, we say that every F-contraction T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

Thus, every F-contraction is a continuous mapping.

- Also, it is easy to see that every ordinary contraction mapping is an F-contraction with $F_1(\alpha) = \ln \alpha$.
- If we consider $F_2(\alpha) = \alpha + \ln \alpha$. Then each self mappings T on a metric space (X, d) satisfying (2) is an F_2-contraction such that

$$\frac{d(Tx, Ty)}{d(x, y)} e^{d(Tx,Ty)-d(x,y)} \leq e^{-\tau}, \text{ for all } x, y \in X, Tx \neq Ty. \tag{3}$$

- Also, Wardowski concluded that if $F, G \in \mathcal{F}$ with $F(\alpha) \leq G(\alpha)$ for all $\alpha > 0$ and $H = G - F$ is nondecreasing, then every F-contraction T is an G-contraction.

He noted that for the mappings $F_1(\alpha) = \ln \alpha$ and $F_2(\alpha) = \alpha + \ln \alpha$, $F_1 < F_2$ and a mapping $F_2 - F_1$ is strictly increasing. Hence, it obtained that every ordinary contraction satisfies the contractive condition (3). On the other side, the following example, which is Example 2.5 in [1], shows that the mapping T is not F_1-contraction (ordinary contraction), but still is an F_2-contraction.

Example

Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$. Define the mapping $T : X \to X$ by $T(x_1) = x_1$ and $T(x_n) = x_{n-1}$ for $n > 1$. Since $\lim_{n\to\infty} \frac{d(Tx_n, Tx_1)}{d(x_n, x_1)} = 1$, the mapping T is not ordinary contraction. But after the some calculation we can see that T is an F_2-contraction with $F_2(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$.

[1] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012, 2012:94, 6 pp.

Thus, the following theorem, which was given by Wardowski, is a proper generalization of Banach Contraction Principle.

Theorem (Wardowski [1])

Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction Then T has a unique fixed point in X.

[1] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012, 2012:94, 6 pp.

By combining the ideas of Wardowski, Ćirić and Berinde, the following results for single valued mappings are obtained.

Theorem (Mınak et al. [2], Wardowski-Van Dung [3])

Let (X, d) be a complete metric space and $T : X \to X$ be a Ćirić type generalized F-contraction, that is, there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(M(x, y))$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}.$$

If T or F is continuous, then T has a unique fixed point in X.

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[2] G. Mınak, A. Helvacı and I. Altun, Ćirić type generalized F-contractions on complete metric space and fixed point results, Filomat, Accepted.

[3] D. Wardowski and N. Van Dung, Fixed points of F-weak contractions on complete metric spaces, Demonstratio Mathematica, 47 (1) (2014), 146-155.

Theorem (Mınak et al. [2])

Let (X, d) be a complete metric space and $T : X \to X$ be an almost F-contraction, that is, there exist $F \in \mathcal{F}$, $\tau > 0$ and $\lambda \geq 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y) + \lambda d(y, Tx))$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$ Then T has a fixed point in X.

We can find some detailed information about Ćirić type generalized F-contractions, almost F-contractions and some counter examples in [2,3].

[2] G. Mınak, A. Helvacı and I. Altun, Ćirić type generalized F-contractions on complete metric space and fixed point results, Filomat, Accepted.

[3] D. Wardowski and N. Van Dung, Fixed points of F-weak contractions on complete metric spaces, Demonstratio Mathematica, 47 (1) (2014), 146-155.

Fixed point theory for multivalued maps

In this section, we recall some fundamental fixed point theorems for multivalued mappins on a complete metric space. Let (X, d) be a metric space.

- $P(X)$ denotes the family of all nonempty subsets of X,
- $C(X)$ denotes the family of all nonempty, closed subsets of X,
- $CB(X)$ denotes the family of all nonempty, closed and bounded subsets of X,
- $K(X)$ denotes the family of all nonempty compact subsets of X.
- It is clear that
$$K(X) \subseteq CB(X) \subseteq C(X) \subseteq P(X).$$
- For $A, B \in C(X)$, let
$$H(A, B) = \max\left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\},$$
where $d(x, B) = \inf \{d(x, y) : y \in B\}$. Then H is called generalized Pompeiu-Hausdorff distance on $C(X)$.
- It is well known that H is a metric on $CB(X)$, which is called Pompeiu-Hausdorff metric induced by d.

In 1969, Nadler first extended the Banach contraction principle to multivalued mappings, Nadler [4] first initiated the study of fixed point theorems for multivalued contraction mappings.

Theorem (Nadler [4])

Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued contraction, that is, there there exists $L \in [0, 1)$ such that

$$H(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$. Then T has a fixed point in X

Then many researchers studied on fixed points of multivalued contractive mappings, which some important of them as follows:

[4] S.B. Nadler, Multi-valued contraction mappings, Pacific J. Math., 30 (1969), 475-488.

Theorem (Reich [5])

Let (X, d) *be a complete metric space and* $T : X \to K(X)$. *Assume that there exists a map* $\varphi : (0, \infty) \to (0, 1)$ *such that*

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s > 0;$$

and

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y).$$

for all $x, y \in X$ *with* $x \neq y$ *Then* T *has a fixed point in* X

In [6], Reich asked the question as if the above theorem is also true for the map $T : X \to CB(X)$. The partial affirmative answer was given by Mizoguchi and Takahashi [7]. They proved the following theorem.

[5] S. Reich, Fixed points of contractive functions, Boll. Unione Mat. Ital. 5 (1972), 26-42.

[6] S. Reich, Some fixed point problems, Atti Acad. Naz. Lincei 57 (1974), 194-198.

[7] N. Mizoguchi and W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl.,141 (1989), 177-188.

Theorem (Mizoguchi-Takahashi [7])

Let (X, d) be a complete metric space and $T : X \to CB(X)$. Assume that there exists a map $\varphi : (0, \infty) \to (0, 1)$ such that

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s \geq 0;$$

and

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y).$$

for all $x, y \in X$ with $x \neq y$ Then T has a fixed point in X

In [8] Suzuki gave a simple proof of Mizoguchi Takahashi fixed point theorem and also an example to show that it is a real generalization of Nadler's.

[7] N. Mizoguchi and W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl.,141 (1989), 177-188.

[8] T. Suzuki, Mizoguchi Takahashi's fixed point theorem is a real generalization of Nadler's, J. Math. Anal. Appl. 340 (2008), 752 755.

Following the above results, Berinde and Berinde [9] introduced a general class of multivalued contractions and proved the following fixed point theorems:

Theorem (Berinde-Berinde [9])

Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued almost contraction, that is, there exist two constants $\delta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx) \qquad (4)$$

for all $x, y \in X$ Then T has a fixed point in X

[9] M. Berinde and V. Berinde, On a general class of multi-valued weakly Picard mappings, J. Math. Anal. Appl., 326 (2007), 772-782.

Theorem (Berinde-Berinde [9])

Let (X, d) *be a complete metric space and* $T : X \to CB(X)$ *be a multivalued nonlinear almost contraction, that is, there exist a constant* $L \geq 0$ *and a function* $\varphi : [0, \infty) \to [0, 1)$ *satisfying*

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s \geq 0, \tag{5}$$

such that

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx) \tag{6}$$

for all $x, y \in X$ *Then* T *has a fixed point in* X

Theorem

A function $\varphi : [0, \infty) \to [0, 1)$ *satisfying (5) is called Mizoguchi-Takahashi function (MT-function) in the literature*

[9] M. Berinde and V. Berinde, On a general class of multi-valued weakly Picard mappings, J. Math. Anal. Appl., 326 (2007), 772-782.

On the other hand, without using the Pompeiu-Hausdorff metric H, many fixed point results for multivalued mappings were obtained. Here we will mention some important of them. For the sake of conformity we denote a set

$$I_b^x = \{y \in Tx : bd(x,y) \leq d(x,Tx)\},$$

where b is a real constant and T is a multivalued mapping on a metric space X. Note that the mapping T is defined from X to $C(X)$ in the following three theorems.

Theorem (Feng-Liu [10])

Let (X,d) be a complete metric space and $T : X \to C(X)$. Assume that the following conditions hold:
(i) the map $x \to d(x,Tx)$ is lower semi-continuous;
(ii) there exist $b, c \in (0,1)$ with $c < b$ such that for any $x \in X$ there is $y \in I_b^x$ satisfying

$$d(y,Ty) \leq cd(x,y).$$

Then T has a fixed point in X

[10] Y. Feng and S. Liu, Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings, J. Math. Anal. Appl., 317 (2006), 103-112.

Then Klim and Wardowski [11] generalized the Feng-Liu's result as follows:

Theorem (Klim-Wardowski [11])

Let (X, d) be a complete metric space and $T : X \to C(X)$. Assume that the following conditions hold:
(i) the map $x \to d(x, Tx)$ is lower semi-continuous;
(ii) there exists $b \in (0, 1)$ and a function $\varphi : [0, \infty) \to [0, b)$ satisfying

$$\limsup_{t \to s^+} \varphi(t) < b, \ \forall s \geq 0$$

and for any $x \in X$, there is $y \in I_b^x$ satisfying

$$d(y, Ty) \leq \varphi(d(x, y))d(x, y).$$

Then T has a fixed point in X

[11] D. Klim and D. Wardowski, Fixed point theorems for set-valued contractions in complete metric spaces, J. Math. Anal. Appl., 334 (2007), 132-139.

Considering the same direction, in 2009, Ćirić [12] introduced new multivalued nonlinear contractions and established a few nice fixed point theorems for such mappings, one of them is as follows:

Theorem (Ćirić [12])

Let (X, d) be a complete metric space and $T : X \to C(X)$ Assume that the following conditions hold:
(i) the map $x \to d(x, Tx)$ is lower semi-continuous;
(ii) there exists a function $\varphi : [0, \infty) \to [a, 1)$, $0 < a < 1$, satisfying

$$\limsup_{t \to s^+} \varphi(t) < 1, \ \forall s \geq 0;$$

(iii) for any $x \in X$, there is $y \in Tx$ satisfying

$$\sqrt{\varphi(d(x, Tx))}d(x, y) \leq d(x, Tx)$$

and

$$d(y, Ty) \leq \varphi(d(x, Tx))d(x, y).$$

Then T has a fixed pointin X

[12] Lj. B. Ćirić, Multi-valued nonlinear contraction mappings, Nonlinear Anal., 71 (2009), 2716-2723.

Analyzing the proofs of above all theorems, we can observe that the mentioned mappings on complete metric spaces are **multivalued weakly Picard (MWP) operators**. We know that, a multivalued map T on a metric space is MWP operator if there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, converges to a fixed point of T.

Multivalued F-contractions

In this section we consider the Wardowski's technique for multivalued mappings.

Definition

Let (X, d) be a metric space and $T : X \to CB(X)$. Then T is said to be a multivalued F-contraction if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(H(Tx, Ty)) \leq F(d(x, y)) \tag{7}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

When we consider $F(\alpha) = \ln \alpha$, we can say that every multivalued contraction is also multivalued F-contraction.

Theorem (Altun et al. [13])

(Theorem MF1)Let (X, d) be a complete metric space and $T : X \to K(X)$ be a multivalued F-contraction, then T has a fixed point in X.

[13] I. Altun, G. Mınak and H. Dağ, Multivalued F-contractions on complete metric space, Journal of Nonlinear and Convex Analysis, Accepted.

In the proof of this theorem we use the following important property: Let A be a compact subset of a metric space (X, d) and $x \in X$, then there exists $a \in A$ such that $d(x, a) = d(x, A)$.

Proof.

Let $x_0 \in X$. As Tx is nonempty for all $x \in X$, we can choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and so the proof is complete. Let $x_1 \notin Tx_1$. Then, since Tx_1 is closed, $d(x_1, Tx_1) > 0$. On the other hand, from $d(x_1, Tx_1) \leq H(Tx_0, Tx_1)$ and $(F1)$

$$F(d(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)).$$

From (7), we can write that

$$F(d(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)) \leq F(d(x_1, x_0)) - \tau \tag{8}$$

Since Tx_1 is compact, we obtain that there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = d(x_1, Tx_1)$. Then, from (8)

$$F(d(x_1, x_2)) \leq F(H(Tx_0, Tx_1)) \leq F(d(x_1, x_0)) - \tau$$

If we continue recursively, then we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$F(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n-1})) - \tau \tag{9}$$

for all $n = 0, 1, 2, \cdots$. □

Proof.

If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} \in Tx_{n_0}$, then x_{n_0} is a fixed point of T and so the proof is complete. Thus, suppose that for every $n \in \mathbb{N}$, $x_n \notin Tx_n$. Denote $a_n = d(x_n, x_{n+1})$, for $n = 0, 1, 2, \cdots$. Then $a_n > 0$ for all $n \in \mathbb{N}$ and, using (9), the following holds:

$$F(a_n) \leq F(a_{n-1}) - \tau \leq F(a_{n-2}) - 2\tau \leq \cdots \leq F(a_0) - n\tau. \tag{10}$$

From (10), we get $\lim_{n\to\infty} F(a_n) = -\infty$. Thus, from $(F2)$, we have

$$\lim_{n\to\infty} a_n = 0.$$

From $(F3)$ there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0.$$

By (10), the following holds for all $n \in \mathbb{N}$

$$a_n^k F(a_n) - a_n^k F(a_0) \leq -a_n^k n\tau \leq 0. \tag{11}$$

Letting $n \to \infty$ in (11), we obtain that

$$\lim_{n\to\infty} n a_n^k = 0. \tag{12}$$

□

Proof.

From (12), there exits $n_1 \in \mathbb{N}$ such that $na_n^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$a_n \leq \frac{1}{n^{1/k}}. \tag{13}$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (13), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \\ &= \sum_{i=n}^{m-1} a_i \leq \sum_{i=n}^{\infty} a_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}} \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$, passing to limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is,$\lim_{n\to\infty} x_n = z$. Taking into accout (7), we have

$$d(x_{n+1}, Tz) \leq H(Tx_n, Tz) \leq d(x_n, z).$$

Passing to limit $n \to \infty$, we obtain $d(z, Tz) = 0$.Thus, we get $z \in \overline{Tz} = Tz$. □

Note that in this theorem, Tx is compact for all $x \in X$. Thus, we can present the following problem: Can we replace $CB(X)$ instead of $K(X)$ in this theorem. In the following example shows that this is not possible with the same conditions.

Example (Atun et al. [14])

Let $X = [0, 1]$ and

$$d(x, y) = \begin{cases} 0 & , \quad x = y \\ 1 + |x - y| & , \quad x \neq y \end{cases},$$

then it is clear that (X, d) is complete metric space, which is also bounded. Since τ_d is discrete topology, all subsets of X are closed. Therefore all subsets of X are closed and bounded. Define $T : X \to CB(X)$ as:

$$Tx = \begin{cases} A & , \quad x \in B \\ B & , \quad x \in A \end{cases},$$

where A is the set of all rational numbers in X and B is the set of all irrational numbers in X. Therefore T has no fixed point.

[14] I. Altun, G. Durmaz, G. Mınak and S. Romaguera, Multivalued almost F-contractions on complete metric spaces, Filomat, Accepted.

Example

Now, define $F : (0, \infty) \to \mathbb{R}$ by

$$F(\alpha) = \begin{cases} \ln \alpha & , \quad \alpha \leq 1 \\ \alpha & , \quad \alpha > 1 \end{cases},$$

then we can see that $F \in \mathcal{F}$. After the some calculation, we can see that T is multivalued F-contraction. Consequently all conditions of the above theorem except for Tx is compact are satisfied, but T has no fixed point.

- Here, if we consider the following condition on F, we can take $CB(X)$ instead of $K(X)$ in this theorem.

$(F4)$ $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

- Note that if F satisfies $(F1)$, then it satisfies $(F4)$ if and only if it is right continuous.
- We denote by $\mathcal{F}_*$ be the set of all functions F satisfying $(F1)$-$(F4)$. For example, let $F(\alpha) = \ln \alpha$ for $\alpha \leq 1$ and $F(\alpha) = 2\alpha$ for $\alpha > 1$, then it is clear that $F \in \mathcal{F} \backslash \mathcal{F}_*$.

Theorem (Altun et al. [13])

(Theorem MF2)Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued F-contraction with $F \in \mathcal{F}_*$, then T has a fixed point in X.

Proof.

As in the proof of the above theorem we can write that

$$F(d(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)) \leq F(d(x_1, x_0)) - \tau. \tag{14}$$

From $(F4)$ we obtain (note that $d(x_1, Tx_1) > 0$)

$$F(d(x_1, Tx_1)) = \inf_{y \in Tx_1} F(d(x_1, y)),$$

and so from (14) we have

$$\inf_{y \in Tx_1} F(d(x_1, y)) \leq F(d(x_1, x_0)) - \tau < F(d(x_1, x_0)) - \frac{\tau}{2}. \tag{15}$$

Then, from (15) there exists $x_2 \in Tx_1$ such that

$$F(d(x_1, x_2)) \leq F(d(x_1, x_0)) - \frac{\tau}{2}.$$

The rest of the proof can be completed as in the proof of previous theorem. □

In the light of the Wardowski's example, we can give the following. This example shows that T is a multivalued F-contraction but it is not multivalued contraction.

Example (Altun et al. [13])

Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define the mapping $T : X \to CB(X)$ by the formulae:

$$Tx = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \{x_1, x_2, \cdots, x_{n-1}\} & , \quad x = x_n \end{cases} .$$

Then T is a multivalued F-contraction with respect to $F(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$, but it is not multivalued contraction.

[13] I. Altun, G. Mınak and H. Dağ, Multivalued F-contractions on complete metric space, Journal of Nonlinear and Convex Analysis, Accepted.

Now we consider τ as a function of $d(x, y)$ in the definition of multivalued F-contraction and define a new concept of multivalued nonlinear F-contraction. Then we give some fixed point results for mappings of this type on complete metric spaces. In a special case, we obtain the Mizoguchi-Takahashi fixed point theorem.

Definition

Let (X, d) be a metric space, $T : X \to CB(X)$ and $\tau : (0, \infty) \to (0, \infty)$ be two mappings. Given $F \in \mathcal{F}$, we say that T is a multivalued nonlinear F-contraction such that

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y)) \quad (16)$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

Theorem (Olgun et al. [15])

(Theorem MF3)Let (X,d) *be a complete metric space and* $T : X \to K(X)$ *be a multivalued nonlinear* F*-contraction If* τ *satisfies*

$$\liminf_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0,$$

then T *has a fixed pointin* X

By considering the condition $(F4)$ we can prove the following:

Theorem (Olgun et al. [15])

(Theorem MF4)Let (X,d) *be a complete metric space and* $T : X \to CB(X)$ *be a multivalued nonlinear* F*-contraction with* $F \in \mathcal{F}_*$ *If* τ *satisfies*

$$\liminf_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0,$$

then T *has a fixed point in* X

[15] M. Olgun, G. Mınak and I. Altun, A new approach to Mizoguchi-Takahashi type fixed point theorem, Journal of Nonlinear and Convex Analysis, Accepted.

If we take $F(\alpha) = \ln \alpha$ in the last theorem, we have the following corollaries.

Corollary

Let (X, d) *be a complete metric space Suppose that* $T : X \to CB(X)$ *satisfies*

$$H(Tx, Ty) \leq e^{-\tau(d(x,y))} d(x, y),$$

for all $x, y \in X$, $x \neq y$, *where* $\tau : (0, \infty) \to (0, \infty)$ *satisfying* $\liminf_{t \to s^+} \tau(t) > 0$ *for all* $s \geq 0$. *Then* T *has a fixed point in* X

Corollary (Mizoguchi-Takahashi)

Let (X, d) *be a complete metric space Suppose that* $T : X \to CB(X)$ *satisfies*

$$H(Tx, Ty) \leq \varphi(d(x, y)) d(x, y),$$

for all $x, y \in X$, $x \neq y$, *where* $\varphi : (0, \infty) \to (0, 1)$ *satisfying* $\limsup_{t \to s^+} \varphi(t) < 1$ *for all* $s \geq 0$. *Then* T *has a fixed point in* X

Proof.

Define $\tau(t) = -\ln \varphi(t)$. Since $\limsup_{t \to s^+} \varphi(t) < 1$ for all $s \geq 0$, then $\liminf_{t \to s^+} \tau(t) > 0$ for all $s \geq 0$. Therefore, by the previous corollary, the proof is complete. □

By considering the almost contraction method, we introduce some new concept of multivalued almost F-contraction and multivalued nonlinear almost F-contraction. Then we give some fixed point results for mappings of these type on complete metric spaces. In a special case, we obtain the Berinde-Berinde fixed point theorem.

Definition

Let (X, d) be a metric space and $T : X \to CB(X)$. We say that T is a multivalued almost F-contraction if there exist $F \in \mathcal{F}$, $\tau > 0$ and $\lambda \geq 0$ such that

$$\tau + F(H(Tx, Ty)) \leq F((d(x, y) + \lambda d(y, Tx)) \tag{17}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

Theorem (Altun et al. [14])

(Theorem MF5)Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued almost F-contraction with $F \in \mathcal{F}_*$, then T is an MWP operator

[14] I. Altun, G. Durmaz, G. Mınak and S. Romaguera, Multivalued almost F-contractions on complete metric spaces, Filomat, Accepted.

Remark

Taking into account the example in [14], we can say that the condition $(F4)$ on F can not be removed in this theorem. But, if we take $T : X \to K(X)$, we can remove the condition $(F4)$ on F.

Remark

If there exist $\delta \in (0,1)$ and $L \geq 0$ satisfying (4) (multivalued almost contraction condition), then (17) is satisfied with $F(\alpha) = \ln \alpha$, $\tau = -\ln \delta$ and $\lambda = \frac{L}{\delta}$. Therefore, the result of Berinde-Berinde is a special case of this theorem.

Remark

If there exist $\tau > 0$ and $F \in \mathcal{F}_*$ satisfying (7) (multivalued F-contraction condition), then (17) is satisfied with $\lambda = 0$. Therefore, the result of Altun et al is a special case of this theorem.

[14] I. Altun, G. Durmaz, G. Mınak and S. Romaguera, Multivalued almost F-contractions on complete metric spaces, Filomat, Accepted.

Now we consider the following two examples.

Example (Altun et al. [14])

Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x,y) = |x-y|$. Then (X,d) is a complete metric space. Define a mapping $T : X \to CB(X)$ by:

$$Tx = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \{x_1, x_2, \cdots, x_{n-1}\} & , \quad x = x_n \end{cases}.$$

Then, as shown in previous example, T is multivalued almost F-contraction with respect to $F(\alpha) = \alpha + \ln \alpha$, $\tau = 1$ and $\lambda \geq 0$. Thus T is an MWP operator. But it can be shown that T is not multivalued almost contraction.

[14] I. Altun, G. Durmaz, G. Mınak and S. Romaguera, Multivalued almost F-contractions on complete metric spaces, Filomat, Accepted.

Example (Altun et al. [14])

Let $X = [0,1] \cup \{2,3\}$ and $d(x,y) = |x-y|$, then (X,d) is complete metric space. Define a map $T : X \to CB(X)$,

$$Tx = \begin{cases} \left[\frac{1-x}{3}, \frac{1-x}{2}\right] & , \quad x \in [0,1] \\ \{x\} & , \quad x \in \{2,3\} \end{cases}.$$

Since $H(T2, T3) = 1 = d(2,3)$, then for all $F \in \mathcal{F}$ and $\tau > 0$ we have

$$\tau + F(H(T2, T3)) > F(d(2,3)).$$

Therefore, T is not a multivalued F-contraction. But after the some calculation, we can see that T is multivalued almost F-contraction with $F(\alpha) = \ln \alpha$, $\tau = \ln 2$ and $\lambda = 10$.

[14] I. Altun, G. Durmaz, G. Mınak and S. Romaguera, Multivalued almost F-contractions on complete metric spaces, Filomat, Accepted.

By considering τ as a function of $d(x, y)$ in the definition of multivalued almost F-contraction, we give a new concept of multivalued nonlinear almost F-contraction.

Definition

Let (X, d) be a metric space and $T : X \to CB(X)$. We say that T is a multivalued nonlinear almost F-contraction with $F \in \mathcal{F}$ if there exist a constant $\lambda \geq 0$ and a function $\tau : (0, \infty) \to (0, \infty)$ such that

$$\liminf_{t \to s^+} \tau(t) > 0, \text{ for all } s \geq 0 \tag{18}$$

satisfying

$$\tau(d(x, y)) + F(H(Tx, Ty)) \leq F((d(x, y) + \lambda d(y, Tx)) \tag{19}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

Remark

It is clear that every multivalued almost F-contraction is also multivalued nonlinear almost F-contraction

Remark

Every multivalued nonlinear almost contraction is also multivalued nonlinear almost F-contraction with a special F Indeed, let (X, d) be metric space and T be a multivalued nonlinear almost contraction Then, there exist a constant $L \geq 0$ and an MT-function φ satisfying

$$H(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx) \tag{20}$$

for all $x, y \in X$ Define $\beta(t) = \frac{1+\varphi(t)}{2}$, then β is also an MT-function Therefore from (20),

$$H(Tx, Ty) \leq \beta(d(x, y))[d(x, y) + 2Ld(y, Tx)]$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$ Thus, we get

$$-\ln(\beta(d(x, y))) + \ln(H(Tx, Ty)) \leq \ln(d(x, y) + 2Ld(y, Tx)) \tag{21}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$ Now, define $\tau(t) = -\ln \beta(t)$ Since β is an MT-function, then

$$\liminf_{t \to s^+} \tau(t) > 0 \text{ for all } s \geq 0.$$

Therefore, from (21), T is a multivalued nonlinear almost F-contraction with $F(\alpha) = \ln \alpha$, $\lambda = 2L$ and $\tau(t) = -\ln\left(\frac{1+\varphi(t)}{2}\right)$

Theorem (Mınak et al. [16])

(Theorem MF6)Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued nonlinear almost F-contraction with $F \in \mathcal{F}_*$, then T is an MWP operator

Example (Mınak et al. [16])

Consider the complete metric space (X, d), where $X = \{\frac{1}{n^2} : n \in \mathbb{N},\ n \geq 2\} \cup \{0\}$ and $d : X \times X \to [0, \infty)$ is given by $d(x, y) = |x - y|$. Define $T : X \to CB(X)$ by

$$Tx = \begin{cases} \left\{0, \frac{1}{(n+1)^2}\right\} & , \quad x = \frac{1}{n^2}, n > 2 \\ \{x\} & , \quad x = \left\{0, \frac{1}{4}\right\} \end{cases}.$$

Since $H(T0, T\frac{1}{4}) = \frac{1}{4} = d(0, \frac{1}{4})$, then for all $F \in \mathcal{F}_*$ and $\tau : (0, \infty) \to (0, \infty)$ satisfying inequality (18), we have

$$\tau(d(0, \frac{1}{4})) + F(H(T0, T\frac{1}{4})) > F(d(0, \frac{1}{4})).$$

[16] G. Mınak, I. Altun and S. Romaguera, Recent developments about multivalued weakly Picard operators, Submitted.

Example

Therefore T is not multivalued nonlinear F-contraction. We can also see that T is not multivalued nonlinear almost contraction. But T is multivalued nonlinear almost F-contraction with $\lambda = 1$, $\tau = \ln \frac{100}{81}$ and

$$F(\alpha) = \begin{cases} \frac{\ln \alpha}{\sqrt{\alpha}} & , \quad 0 < \alpha < e^2 \\ \frac{2\alpha}{e^3} & , \quad \alpha \geq e^2 \end{cases} .$$

Fixed point results without using Pompeiu-Hausdorff metric

Let $T : X \to P(X)$ be a multivalued map, $F \in \mathcal{F}$ and $\sigma \geq 0$. For $x \in X$ with $d(x, Tx) > 0$, define the set $F_\sigma^x \subseteq X$ as

$$F_\sigma^x = \{y \in Tx : F(d(x,y)) \leq F(d(x,Tx)) + \sigma\}.$$

We need to consider the following cases:

- If $T : X \to K(X)$, then for all $\sigma \geq 0$ and $x \in X$ with $d(x, Tx) > 0$, we have $F_\sigma^x \neq \emptyset$. Indeed, since Tx is compact, for every $x \in X$ we have $y \in Tx$ such that $d(x,y) = d(x,Tx)$. Therefore, for every $x \in X$ with $d(x,Tx) > 0$, we have $F(d(x,y)) = F(d(x,Tx))$. Thus $y \in F_\sigma^x$ for all $\sigma \geq 0$.
- If $T : X \to C(X)$, then F_σ^x may be empty for some $x \in X$ and $\sigma > 0$. For example, let $F(\alpha) = \ln \alpha$ for $\alpha \leq 1$ and $F(\alpha) = 2\alpha$ for $\alpha > 1$ and let $X = \{0\} \cup (1,2)$ with the usual metric. Define $T : X \to C(X)$ by $T0 = (1,2)$ and $Tx = \{0\}$ for $x \in (1,2)$. Then, for $x = 0$ we have (note that $d(0, T0) = 1 > 0$)

$$\begin{aligned} F_1^0 &= \{y \in T0 : F(d(0,y)) \leq F(d(0,T0)) + 1\} \\ &= \{y \in (1,2) : F(y) \leq F(1) + 1\} \\ &= \{y \in (1,2) : 2y \leq 1\} \\ &= \emptyset. \end{aligned}$$

- If $T : X \to C(X)$ (even if $T : X \to P(X)$) and $F \in \mathcal{F}_*$, then for all $\sigma > 0$ and $x \in X$ with $d(x, Tx) > 0$, we have $F_\sigma^x \neq \emptyset$. Indeed, by (F4), we have

$$\begin{aligned} F_\sigma^x &= \{y \in Tx : F(d(x,y)) \leq F(d(x,Tx)) + \sigma\} \\ &= \{y \in Tx : F(d(x,y)) \leq F(\inf\{d(x,y) : y \in Tx\}) + \sigma\} \\ &= \{y \in Tx : F(d(x,y)) \leq \inf\{F(d(x,y)) : y \in Tx\} + \sigma\} \\ &\neq \emptyset. \end{aligned}$$

By considering the above facts we give the following theorems:

Theorem (Mınak et al. [17])

(Theorem MF7)Let (X, d) *be a complete metric space,* $T : X \to K(X)$ *be a multivalued map and* $F \in \mathcal{F}$ *If there exists* $\tau > 0$ *such that for any* $x \in X$ *with* $d(x, Tx) > 0$*, there exists* $y \in F_\sigma^x$ *satisfying*

$$\tau + F(d(y, Ty)) \leq F(d(x, y)),$$

then T *has a fixed point in* X *provided* $\sigma < \tau$ *and* $x \to d(x, Tx)$ *is lower semi-continuous*

Theorem (Mınak et al. [17])

(Theorem MF8)Let (X, d) *be a complete metric space,* $T : X \to C(X)$ *and* $F \in \mathcal{F}_*$ *If there exists* $\tau > 0$ *such that for any* $x \in X$ *with* $d(x, Tx) > 0$*, there exists* $y \in F_\sigma^x$ *satisfying*

$$\tau + F(d(y, Ty)) \leq F(d(x, y))$$

then T *has a fixed point in* X *provided* $0 < \sigma < \tau$ *and* $x \to d(x, Tx)$ *is lower semi-continuous*

[17] G. Mınak, M. Olgun and I. Altun, A new approach to fixed point theorems for multivalued contractive maps Carpathian Journal of Mathematics, Accepted.

Corollary (Feng-Liu)

Let (X,d) be a complete metric space and $T : X \to C(X)$ If there exists $c \in (0,1)$ such that for any $x \in X$, there exists $y \in I_b^x$ $(b \in (0,1))$ satisfying

$$d(y,Ty) \leq cd(x,y),$$

then T has a fixed point in X provided $c < b$ and $x \to d(x,Tx)$ is lower semi-continuous

Remark

Theorem MF7 is a generalization of Theorem MF1 In fact, let T satisfies the conditions of Theorem MF1 Since every multivalued F-contractions are multivalued nonexpansive and every multivalued nonexpansive maps are upper semi-continuous, then T is upper semi-continuous Therefore, the function $x \to d(x,Tx)$ is lower semi-continuous (see the Proposition 4 2 6 of [18]) On the other hand, for any $x \in X$ with $d(x,Tx) > 0$ and $y \in F_\sigma^x$, we have

$$\tau + F(d(y,Ty)) \leq \tau + F(H(Tx,Ty)) \leq F(d(x,y)).$$

Hence T satisfies conditions of Theorem MF7, the existence of a fixed point has been proved There is the similar relation between Theorem MF2 and Theorem MF8

[18] R. P. Agarwal, D. O'Regan and D. R. Sahu, Fixed Point Theory for Lipschitzian-type Mappings with Applications, Springer, New York, 2009.

Example (Mınak et al. [17])

Let $X = \{\frac{1}{2^{n-1}} : n \in \mathbb{N}\} \cup \{0\}$ with the usual metric d, then (X, d) is a complete metric space. Define a mapping $T : X \to C(X)$ as

$$Tx = \begin{cases} \{\frac{1}{2^n}, 1\} & , \quad x = \frac{1}{2^{n-1}} \\ \{0, \frac{1}{2}\} & , \quad x = 0 \end{cases}.$$

Since $H(T\frac{1}{2}, T0) = \frac{1}{2} = d(\frac{1}{2}, 0)$, then for all $F \in \mathcal{F}$ and $\tau > 0$ we have

$$\tau + F(H(T\frac{1}{2}, T0)) > F(d(\frac{1}{2}, 0)).$$

Thus T is not multivalued F-contraction. Therefore Theorem MF1 and Theorem MF2 can not be applied to this example.

On the other hand, it is easy to compute that all conditions of Theorem MF7 and Theorem MF8 are satisfied and so T has a fixed point.

[17] G. Mınak, M. Olgun and I. Altun, A new approach to fixed point theorems for multivalued contractive maps Carpathian Journal of Mathematics, Accepted.

In the following theorem we replace $P(X)$ by $C(X)$, but we need to add an extra condition.

Theorem (Mınak et al. [17])

(Theorem MF9)Let (X,d) be a complete metric space, $T : X \to P(X)$ and $F \in \mathcal{F}_*$ Suppose there exists $\tau > 0$ such that for any $x \in X$ with $d(x,Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying $d(y,Ty) > 0$ and

$$\tau + F(d(y,Ty)) \leq F(d(x,y)).$$

If there exists $x_0 \in X$ with $d(x_0,Tx_0) > 0$ such that for all convergent sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$, we have $T(\lim x_n)$ is closed, then T has a fixed point in X provided $\sigma < \tau$ and $x \to d(x,Tx)$ is lower semi-continuous

Corollary

Let (X,d) be a complete metric space and $T : X \to P(X)$ Suppose there exists $c \in (0,1)$ such that for any $x \in X$ with $d(x,Tx) > 0$ there exists $y \in I_b^x$ $(b \in (0,1))$ satisfying

$$0 < d(y,Ty) \leq cd(x,y). \tag{22}$$

If there exists $x_0 \in X$ with $d(x_0,Tx_0) > 0$ such that for all convergent sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$, we have $T(\lim x_n)$ is closed, then T has a fixed point in X provided $c < b$ and $x \to d(x,Tx)$ is lower semi-continuous

Example (Mınak et al. [17])

Let $X = [0,2]$ with the usual metric. Define $T : X \to P(X)$ as

$$Tx = \begin{cases} (\frac{x}{4}, \frac{x}{2}] & , \quad x \in (0,1] \\ \\ \{\frac{x}{2}\} & , \quad x \in \{0\} \cup (1,2] \end{cases} .$$

Since Tx is not closed for some $x \in X$, both Nadler and Feng-Liu's results can not be applied to this example. On the other hand if we take $\frac{1}{2} \leq c < b$ and $x_0 \in (0,2]$, then all conditions of the above Corollary are satisfied. Therefore T has a fixed point.

[17] G. Mınak, M. Olgun and I. Altun, A new approach to fixed point theorems for multivalued contractive maps Carpathian Journal of Mathematics, Accepted.

Theorem (Altun et al. [19])

(Theorem MF10)Let (X,d) *be a complete metric space,* $T : X \to C(X)$ *and* $F \in \mathcal{F}_*$
Assume that the following conditions hold:
(i) the map $x \to d(x, Tx)$ *is lower semi-continuous;*
(ii) there exist $\sigma > 0$ *and a function* $\tau : (0,\infty) \to (\sigma,\infty)$ *such that*

$$\liminf_{t \to s^+} \tau(t) > \sigma \text{ for all } s \geq 0$$

and for any $x \in X$ *with* $d(x, Tx) > 0$*, there exists* $y \in F_\sigma^x$ *satisfying*

$$\tau(d(x,y)) + F(d(y, Ty)) \leq F(d(x,y)).$$

Then T *has a fixed point*

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[19] I. Altun, G. Mınak and M. Olgun, Fixed points of multivalued nonlinear F-contractions on complete metric spaces, Submitted.

In the following example, we show that there are some multivalued mappings such that Theorem MF10 can be applied but Klim-Wardowski's result can not.

Example

Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define a mapping $T : X \to C(X)$ by the formulae:

$$Tx = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \{x_1, x_{n-1}\} & , \quad x = x_n \end{cases} .$$

Then, since τ_d is discrete topology, the map $x \to d(x, Tx)$ is continuous. Now we claim that the condition (ii) of Klim-Wardowski's is not satisfied. Indeed, let $x = x_n$ for $n > 1$, then $Tx = \{x_1, x_{n-1}\}$. In this case, for all $b \in (0, 1)$, there exists $n_0(b) \in \mathbb{N}$ such that for all $n \geq n_0(b)$, $I_b^{x_n} = \{x_{n-1}\}$. Thus, for $n \geq n_0(b)$ we have

$$d(y, Ty) = n - 1, \ d(x, y) = n.$$

Therefore since $\frac{d(y,Ty)}{d(x,y)} = \frac{n-1}{n}$, we can not find a function $\varphi : [0, \infty) \to [0, b)$ satisfying

$$d(y, Ty) \leq \varphi(d(x, y))d(x, y).$$

On the other hand the condition (ii) of Theorem MF10 is satisfied with $F(\alpha) = \alpha + \ln \alpha$, $\sigma = \frac{1}{2}$and $\tau(t) = \frac{1}{t} + \frac{1}{2}$.

Remark

If we take $K(X)$ instead of $C(X)$ in Theorem MF10, we can remove the condition $(F4)$ on F Further, by taking into account F_σ^x, we can take $\sigma \geq 0$ Therefore, the proof of the following theorem is obvious

Theorem (Altun et al. [19])

(Theorem MF11)Let (X, d) be a complete metric space and $T : X \to K(X)$. Assume that the following conditions hold:
(i) the map $x \to d(x, Tx)$ is lower semi-continuous;
(ii) there exist $\sigma \geq 0$, $F \in \mathcal{F}$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that

$$\liminf_{t \to s^+} \tau(t) > \sigma \text{ for all } s \geq 0$$

and for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying

$$\tau(d(x, y)) + F(d(y, Ty)) \leq F(d(x, y)).$$

Then T has a fixed point

[19] I. Altun, G. Mınak and M. Olgun, Fixed points of multivalued nonlinear F-contractions on complete metric spaces, Submitted.

Theorem (Altun et al. [20])

(Theorem MF12)Let (X, d) *be a complete metric space,* $T : X \to C(X)$ *and* $F \in \mathcal{F}_*$
Assume that the following conditions hold:
(i) the map $x \to d(x, Tx)$ *is lower semi-continuous;*
(ii) there exists a function $\tau : (0, \infty) \to (0, \sigma]$, $\sigma > 0$ *such that*

$$\liminf_{t \to s^+} \tau(t) > 0, \ \forall s \geq 0; \tag{23}$$

(iii) for any $x \in X$ *with* $d(x, Tx) > 0$, *there is* $y \in Tx$ *satisfying*

$$F(d(x, y)) \leq F(d(x, Tx)) + \frac{\tau(d(x, Tx))}{2} \tag{24}$$

and

$$\tau(d(x, Tx)) + F(d(y, Ty)) \leq F(d(x, y)). \tag{25}$$

Then T *is a MWP operator*

[20] I. Altun, M. Olgun and G. Mınak, On a new class of multivalued weakly Picard operators on complete metric spaces, Taiwanese Journal of Mathematics, Accepted.

Theorem (Altun et al. [20])

(Theorem MF13)Let (X, d) *be a complete metric space,* $T : X \to K(X)$ *and* $F \in \mathcal{F}$
Assume that the following conditions hold:
(i) the map $x \to d(x, Tx)$ *is lower semi-continuous;*
(ii) there exists a function $\tau : (0, \infty) \to (0, \sigma]$, $\sigma > 0$ *such that*

$$\liminf_{t \to s^+} \tau(t) > 0, \ \forall s \geq 0;$$

(iii) for any $x \in X$ *with* $d(x, Tx) > 0$, *there is* $y \in Tx$ *satisfying*

$$F(d(x, y)) \leq F(d(x, Tx)) + \frac{\tau(d(x, Tx))}{2}$$

and

$$\tau(d(x, Tx)) + F(d(y, Ty)) \leq F(d(x, y)).$$

Then T *is a MWP operator*

[20] I. Altun, M. Olgun and G. Mınak, On a new class of multivalued weakly Picard operators on complete metric spaces, Taiwanese Journal of Mathematics, Accepted.

Taking into account our results, T is a MWP operator in the following nontrivial example. We also show that all mentioned theorems except for Theorems MF12 and MF13 can not be applied to this example.

Example

Let $X = \{\frac{1}{n^2} : n \in \mathbb{N}\} \cup \{0\}$ and $d(x,y) = |x-y|$, then (X,d) is complete metric space. Let $T : X \to CB(X)$ be defined by

$$Tx = \begin{cases} \left\{0, \frac{1}{(n+1)^2}\right\} & , \quad x = \frac{1}{n^2} \\ \{x\} & , \quad x \in \{0,1\} \end{cases}.$$

It is easy to see that

$$d(x,Tx) = \begin{cases} 0 & , \quad x \in \{0,1\} \\ \frac{2n+1}{n^2(n+1)^2} & , \quad x = \frac{1}{n^2}, \ n \geq 2 \end{cases}$$

and it is lower semi-continuous.

Example

Let $\tau(t) = \ln 2$ and $\sigma = 4$, then the condition (ii) of Theorem MF12 is satisfied. We can see that the condition (iii) of Theorem MF12 is satisfied with

$$F(\alpha) = \begin{cases} \frac{\ln \alpha}{\sqrt{\alpha}} & , \quad 0 < \alpha < e^2 \\ \alpha - e^2 + \frac{2}{e} & , \quad \alpha \geq e^2 \end{cases}.$$

Thus all conditions of Theorem MF12 are satisfied and so T has a fixed point in X. On the other hand, after some calculation we can see that the other all mentioned theorems can not be applied to this example.

THANK YOU FOR YOUR ATTENTION