Monomial expansions of H_p-functions in infinitely many variables

Andreas Defant

Content

- Where do bounded and holomorphic functions on the infinite dimensional polydisc have a monomial series expansion?

Content

- Where are H_∞-functions in infinitely many variables analytic?

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- Dirichlet series

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The Cauchy approach . . .

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Definition

X a complex normed space

- $f : B_X \longrightarrow \mathbb{C}$ holomorphic

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Definition – today, Fréchet 1915

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- $H_\infty(B_X) :=$
  Banach space of all bounded and holomorphic functions

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Monomial series expansions – the Weierstraß approach **?**

Recall from the theory of finitely many dimensions . . .

the well-known classical fact:

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In infinitely many dimensions?

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How to define monomial series expansions for holomorphic functions on B_{ℓ_∞} ?

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Question

Other z's ? Even all z's ?

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But if we do not let such considerations put us off, then we will be like Siegfried in front of whom the magic fires retreat, and as a reward waits for us the beautiful prize of . . .

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1. $\exists\, 0 \neq f \in H_\infty(B_{\ell_\infty})\ \forall\, z \in B_{\ell_\infty}:\ \ c_\alpha(f) = 0$
2. Toeplitz 1913:
   $\exists\, f \in H_\infty(B_{\ell_\infty})\ \ \exists\, z \in B_{\ell_\infty}:\ \ \sum_\alpha |c_\alpha(f) z^\alpha| = \infty$

$\exists\ 0 \neq f \in H_\infty(B_{\ell_\infty})$: $c_\alpha(f) = 0$ for all α

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Conclusion

Holomorphy and analyticity in infinite dimensions are two substantially different notions . . .

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- $\operatorname{mon} H_\infty(B_{c_0})$

  $:= \left\{ z \in B_{c_0} \mid \forall f \in H_\infty(B_{c_0}) : f(z) = \sum_\alpha c_\alpha(f) z^\alpha \right\}$

  set of monomial convergence of $H_\infty(B_{c_0})$

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- $\operatorname{mon} \mathscr{P}({}^m c_0)$

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Problem

$$\operatorname{mon} H_\infty(B_{c_0}) = \textbf{?}$$

$$\operatorname{mon} \mathscr{P}(^m c_0) = \textbf{?}$$

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Bohr 1913

$$\ell_2 \cap B_{c_0} \subset \operatorname{mon} H_\infty(B_{c_0})$$

How optimal are these results in the scale of ℓ_p's ?

Largest ℓ_p such that $\ell_p \cap B_{c_0} \subset \operatorname{mon} H_\infty(B_{c_0})$ **?**

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$$M := \sup \left\{ 1 \leq p < \infty \,\middle|\, \ell_p \cap B_{c_0} \subset \operatorname{mon} H_\infty(B_{c_0}) \right\}$$

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$$M = 2 \quad \text{and} \quad M_m = \frac{2m}{m-1}$$

An extension based on probabilistic arguments . . .

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D.-Maestre-Prengel 2009

$$\ell_{\frac{2m}{m-1}} \subset \operatorname{mon} \mathscr{P}(^m c_0) \subset \ell_{\frac{2m}{m-1}+\varepsilon} \quad \text{for all } \varepsilon$$

$$\ell_2 \cap B_{c_0} \subset \operatorname{mon} H_\infty(B_{c_0}) \subset \ell_{2+\varepsilon} \cap B_{c_0} \quad \text{for all } \varepsilon$$

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$$\operatorname{mon} H_\infty(B_{c_0}) = \mathbf{?} \ \text{ and } \ \operatorname{mon} \mathscr{P}(^m c_0) = \mathbf{?}$$

Let us explain why we had some hope to improve the results known so far, and in which direction ...

Three reasonable guesses – the main three candidates:

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$$\ell_{2,\infty} := \left\{ z \in \mathbb{C}^{\mathbb{N}} \mid \sup_n z_n^* n^{1/2} < \infty \right\}$$

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3. $$\operatorname{mon} H_\infty(B_{c_0}) = \ell_{2,\log} \cap B_{c_0} \text{ ?}$$
$$\ell_{2,\log} := \left\{ z \in \mathbb{C}^{\mathbb{N}} \mid \sup_n z_n^* \Big(\frac{n}{\log n}\Big)^{1/2} < \infty \right\}$$

The three m-homogeneous candidates:

1. $$\text{mon}\,\mathscr{P}(^m c_0) = \ell_{\frac{2m}{m-1}} \textbf{ ?}$$

2. $$\text{mon}\,\mathscr{P}(^m c_0) = \ell_{\frac{2m}{m-1},\infty} \textbf{ ?}$$

3. $$\text{mon}\,\mathscr{P}(^m c_0) = \ell_{\frac{2m}{m-1},\log} \textbf{ ?}$$

First question:

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The answer is non trivial – we will see in an apparently sophisticated way that:

$$\ell_2 \cap B_{c_0} \subsetneq \operatorname{mon} H_\infty(B_{c_0})$$

$$\ell_{\frac{2m}{m-1}} \subsetneq \operatorname{mon} \mathscr{P}({}^m c_0)$$

Content

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- Dirichlet series
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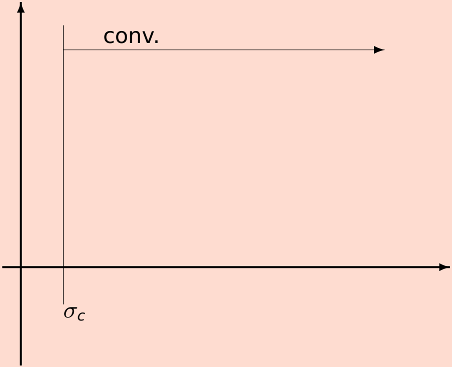

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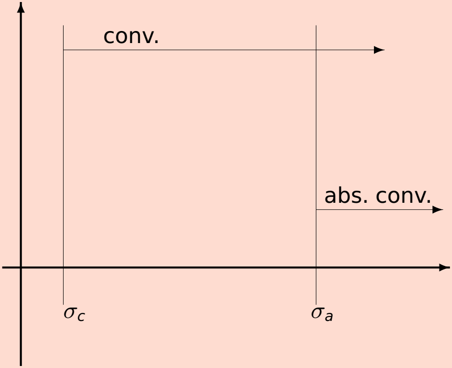

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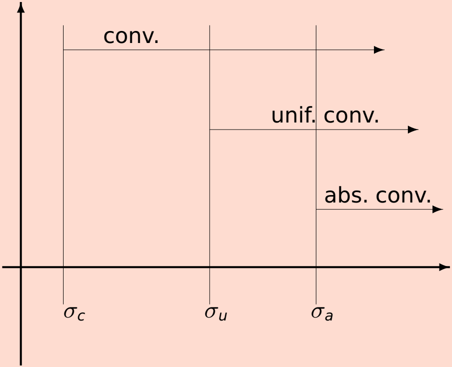

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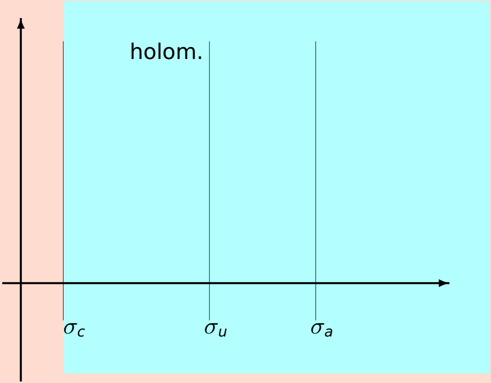

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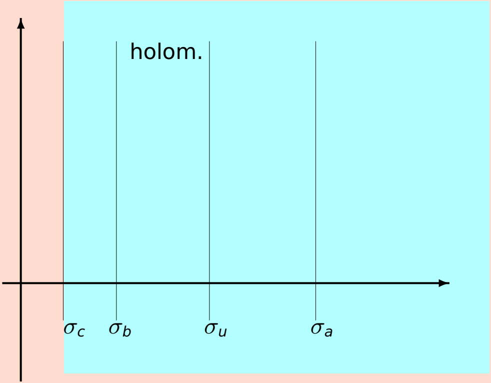

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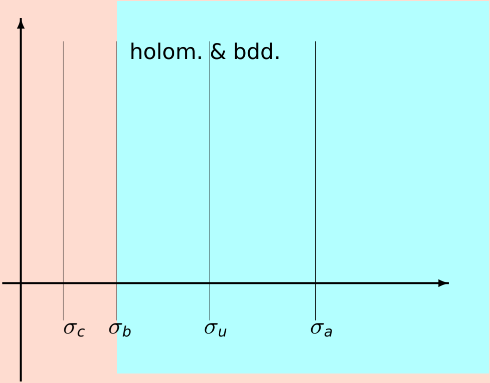
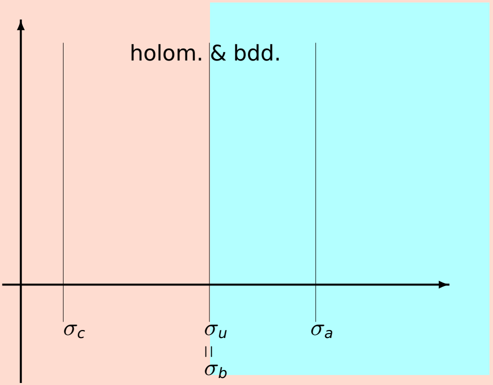


Bohr's fundamental theorem

$$\sigma_u(D) = \sigma_b(D)$$

Convergence of Dirichlet series



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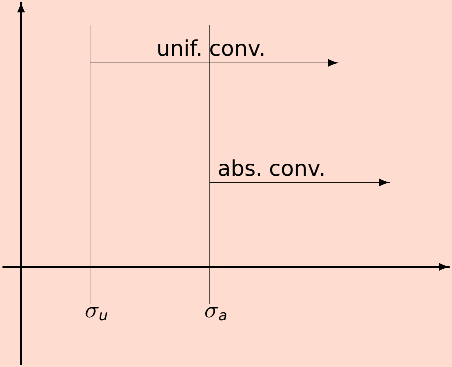

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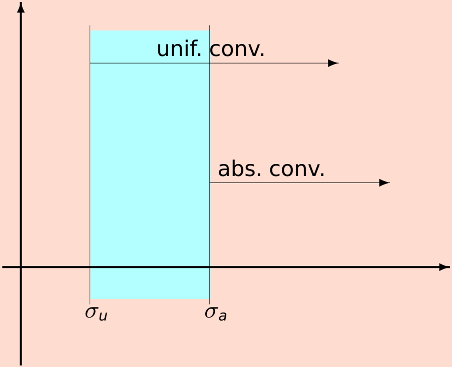

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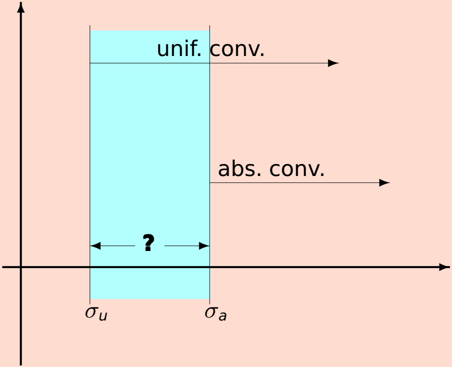

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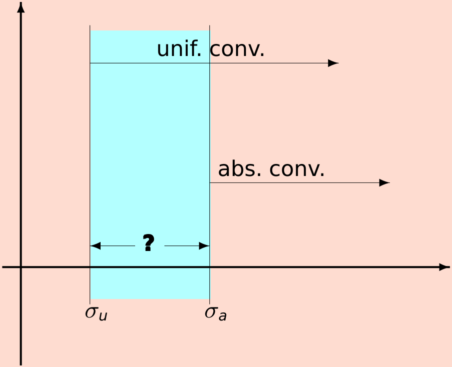

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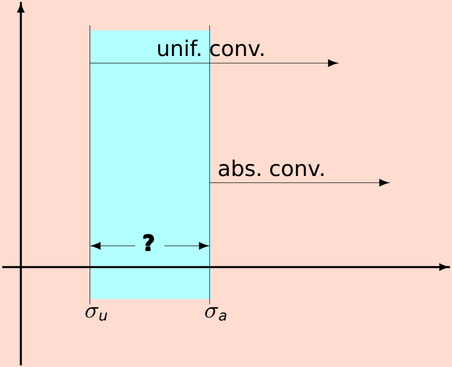
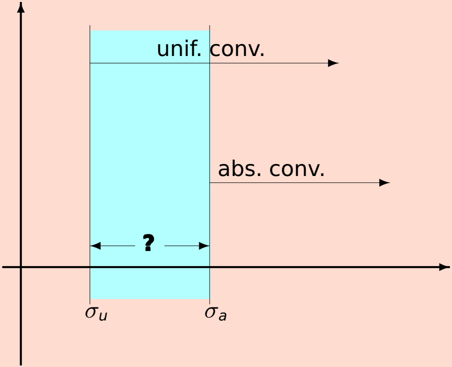
Convergence of Dirichlet series



Bohr's absolute convergence problem

$$S = \textbf{?}$$

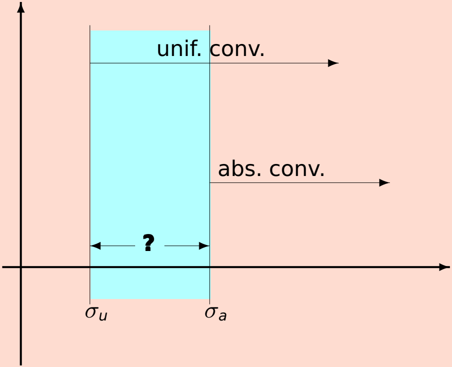
Convergence of Dirichlet series



Bohr-Bohnenblust-Hille Theorem 1913, 1931

$$S = \frac{1}{2}$$

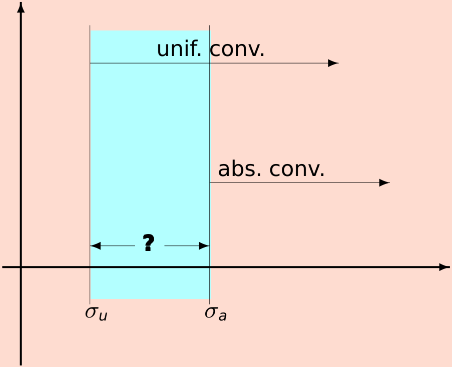
Convergence of Dirichlet series



Definition - graduation of S

$$S_m := \sup\left\{ \sigma_a(D) - \sigma_u(D) \quad : \quad D = \sum_n a_n \frac{1}{n^s} \ m\text{-homogenous} \right\}$$

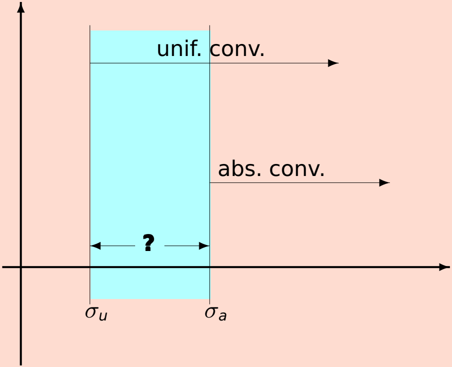
Convergence of Dirichlet series



The Bohr-Bohnenblust-Hille Theorem - graduated

$$S_m = \frac{m-1}{2m}$$

Convergence of Dirichlet series



Again . . .

$$S = \frac{1}{2} \quad \text{and} \quad S_m = \frac{m-1}{2m}$$

How is the absolute convergence problem linked with our problem to determine mon $H_\infty(B_{c_0})$ and mon $\mathscr{P}(^m c_0)$?

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Bohr 1913 – in modern language

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The proof needs the prime number theorem ...

What is the deeper reason for these equalities?

Content

-
-
- Dirichlet series and complex analysis on polydiscs
-
-

$p =$ the sequence of prime numbers: $p_1 < p_2 < p_3 < \dots$

$p^\alpha = p_1^{\alpha_1} \times \cdots \times p_N^{\alpha_N}$ where $\alpha = (\alpha_1, \dots, \alpha_N, 0, \dots) \in \mathbb{N}_0^{(\mathbb{N})}$

A one to one correspondence:

formal power series

$\mathfrak{P}$

A one to one correspondence:

formal power series $\mathfrak{P}$ $\longrightarrow$ Dirichlet series $\mathfrak{D}$

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formal power series		Dirichlet series
$\mathfrak{P}$	$\longrightarrow$	$\mathfrak{D}$
$\sum_\alpha c_\alpha z^\alpha$	$\xrightarrow{a_n = a_{p^\alpha} = c_\alpha}$	$\sum_n a_n \frac{1}{n^s}$

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... in view of Bohr's fundamental theorem these are up to an ε those Dirichlet series which converge uniformly on $[\mathrm{Re} > 0]$.

. . . and for m-homogeneous polynomials:

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$\bigcup$		
$\mathscr{P}(^m c_0)$	$\longrightarrow$	$\mathscr{H}_\infty^m$

Bohr 1913, Hedenmalm-Lindqvist-Seip 1997

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. . . and that our first aim was to explain why

$$\ell_2 \cap B_{c_0} \subsetneqq \operatorname{mon} H_\infty(B_{c_0})$$

$$\ell_{\frac{2m}{m-1}} \subsetneqq \operatorname{mon} \mathscr{P}(^m c_0)$$

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Quantification . . .

Definition

$$S_N := \sup_{a_1,\dots,a_N\in\mathbb{C}} \frac{\sum_{n=1}^{N} |a_n|}{\sup_{t\in\mathbb{R}} \left| \sum_{n=1}^{N} a_n \frac{1}{n^{it}} \right|},$$

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Theorem

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Balasubramanian-Calado-Queffélec 2006
DFOOS 2011

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But: $\Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}}, \dots\Big) \asymp \Big(\frac{1}{\sqrt{n \log n}}\Big)_n \notin \ell_2$ $\square$

Similarly:

$$\ell_{\frac{2m}{m-1}} \subsetneq \text{mon}\, \mathscr{P}({}^m c_0)$$

Balasubramanian-Calado-Queffélec 2006, Maurizi-Queffélec 2010

Let $D = \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \in \mathscr{H}_\infty^m$. Then

$$\sum_{n=1}^{\infty} |a_n| \frac{\log^{\frac{m-1}{2}} n}{n^{\frac{m-1}{2m}}} < \infty,$$

and conversely . . .

Again and again . . . our problem is:

$$\operatorname{mon} H_\infty(B_{c_0}) = \textbf{?} \text{ and } \operatorname{mon} \mathscr{P}(^m c_0) = \textbf{?}$$

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The smallest candidate ℓ_2 describing $\operatorname{mon} H_\infty(B_{c_0})$ is **out**:

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Why seems this sequence space a reasonable candidate?

For each $z \in B_{c_0}$ a simple closed graph argument shows:

$$z \in \operatorname{mon} H_\infty(B_{c_0})$$

$$\Updownarrow$$

$$\exists\, c > 0\, \forall\, f \in H_\infty(B_{c_0}) : \sum_\alpha |c_\alpha(f) z^\alpha| \leq c \|f\|_\infty$$

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Our problem is: Which $z \in B_{c_0}$ satisfy such an inequality?

Bohr writes: . . . *in order to solve this problem we need a deeper understanding of the theory of power series in infinitely many variables . . .*

For each $z \in B_{c_0}$ a simple closed graph argument shows:

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Bohr writes: ... *in order to solve this problem we need a deeper understanding of the theory of power series in infinitely many variables* ...... *much more than I could do* ... *it was in the course of this investigation that I was led to consider a problem concerning power-series in one variable only which seems to be of some interest in itself* ...

Bohr's power series theorem, 1914

- $\forall \ f \in H_\infty(\mathbb{D}): \ \sum_n \left| c_n(f) \left(\frac{1}{3}\right)^n \right| \leq \|f\|_\infty$

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Aizenberg, Boas, Bombieri, Bourgain, ...

Definition – Nth Bohr radius

$$K_N := \text{ best } r \text{ such that } \forall f \in H_\infty(\mathbb{D}^N) \ : \sum_{\alpha \in \mathbb{N}_0^N} |c_\alpha(f)| r^{|\alpha|} \leq \|f\|_\infty$$

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Problem

$$K_N = \textbf{?}$$

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$$K_N \asymp \sqrt{\frac{\log N}{N}}$$

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Theorem, Dineen-Timoney 1989

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Theorem, D.-Frerick 2006

$$\sqrt{\frac{\log N}{N \log\log N}} \prec K_N \prec \sqrt{\frac{\log N}{N}}$$

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Hence the following conjecture seemed tempting:

$$\ell_{2,\log} \cap B_{c_0} = \operatorname{mon} H_\infty(B_{c_0})$$

Content

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- Main results
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First the m-homogeneous case . . .

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No!

Theorem, DFMS 2012

Let $z \in B_{c_0}$.

1. $\limsup_{n\to\infty} \frac{1}{\log n} \sum_{j=1}^{n} z_j^{*2} < \frac{1}{2} \quad \Rightarrow \quad z \in \operatorname{mon} H_\infty(B_{c_0})$.

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Dismissing candidates

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It even turns out:

There is no Banach sequence space X at all such that

$$\operatorname{mon} H_\infty(B_{c_0}) = X \cap B_{c_0}$$

Alternative formulation

$$\frac{1}{\sqrt{2}}\mathbf{B} \ \subset\ \mathrm{mon}\, H_\infty(B_{c_0}) \ \subset\ \overline{\mathbf{B}}\,,$$

where

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An optimistic conjecture:

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Subpolynomial constants in the multilinear BH-inequality have applications to the study of so called XOR-games in quantum information theory . . .

Content

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- Where are H_p-functions in infinitely many variables analytic?

Hardy spaces on the finite dimensional torus $\mathbb{T}^N$

- $H_p(\mathbb{T}^N) := \left\{ f \in L_p(\mathbb{T}^N) \colon \hat{f}(\alpha) = 0 \text{ for all } \alpha \in \mathbb{Z}^N \setminus \mathbb{N}_0^N \right\}$

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A classic (of Rudin?)

For each N and $1 \leq p \leq \infty$ the linear mapping

$$\phi : H_p(\mathbb{T}^N) \to H_p(\mathbb{D}^N), \ \phi(f)(z) := \sum_{\alpha \in \mathbb{N}_0^N} \hat{f}(\alpha) z^\alpha$$

is bijective and isometric.

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Each $f \in H_p(\mathbb{T}^N)$ extends analytically to the interior . . .

Hardy spaces on the infinite dimensional torus $\mathbb{T}^\infty$

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Sets of monomial convergence

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Philosophy . . .

By definition $\operatorname{mon} H_p(\mathbb{T}^\infty)$ describes the largest subset R of B_{c_0} which has the property that each $f \in H_p(\mathbb{T}^\infty)$ extends analytically to R . . .

The case $p = \infty$ corresponds to the preceding one . . .

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Theorem, Cole-Gamelin 1986

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such that for every $f \in H_\infty(\mathbb{T}^\infty)$ and every α

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An immediate consequence

$$\operatorname{mon} H_\infty(\mathbb{T}^\infty) = \operatorname{mon} H_\infty(B_{c_0})$$

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1. $\frac{1}{\sqrt{2}}\mathbf{B} \subset \operatorname{mon} H_\infty(\mathbb{T}^\infty) \quad \subset \overline{\mathbf{B}}$

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Theorem, DFMS 2012

$$\operatorname{mon} H_p^m(\mathbb{T}^\infty) = \begin{cases} \ell_2 \cap \mathbb{D}^{\mathbb{N}} & \text{for } 1 \leq p < \infty \\ \ell_{\frac{2m}{m-1},\infty} \cap \mathbb{D}^{\mathbb{N}} & \text{for } p = \infty \end{cases}$$