Some remarks on cone metric spaces

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In this talk $(E, +)$ will be a real Banach space, P a cone in E with $int(P) \neq \emptyset$ and $\lesssim$ the partial order on E with respect to P.

Definition

(Huang and Zhang [3]). A cone metric space is an ordered 4-tern (X, d, E, P), where X is non-empty set, E is a real Banach space, P is a cone of E with $intP \neq \emptyset$ and $d : X \times X \to P$ is a mapping satisfying

(d1) $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, z) \lesssim d(x, y) + d(y, z)$ for all $x, y, z \in X$.

In this case, we will say that d is a cone metric on (X, E, P).

Remark: Clearly a classical metric space (X, d) is a cone metric space. Indeed, $P = [0, \infty[$ is a cone with $int(P) \neq \emptyset$ in the real Banach space $(\mathbb{R}, |\cdot|)$

Example 1 (Huang and Zhang [3])

Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\} \subseteq \mathbb{R}^2$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$. Then, (X, d, E, P) is a cone metric space

Example 2 (Rezapour and Hamlbarani [7])

Let $E = l_1$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0,$ for all $n \in \mathbb{N}\}$, (X, p) be a metric space and define $d : X \times X \to P$ by

$$d(x, y) = \left\{ \frac{\rho(x, y)}{2^n} \right\}_{n\geq 1}.$$

(X, d, E, P) is a cone metric space.

Definition

(Huang and Zhang [3]). Let $\{x_n\}$ be a sequence in the cone metric space (X, d) and let $x \in X$. Then:

(i) $\{x_n\}$ is called convergent to x if for any $c \in P$ with $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq n_0$.

(ii) $\{x_n\}$ is called Cauchy if for any $c \in P$ with $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$.

(iii) a cone metric (X, d) is called complete if every Cauchy sequence is convergent.

Let (X, d, E, P) be a cone metric space. Defining

$$B(x, r) = \{y \in X : d(x, y) \ll r\}, \text{ where } x \in X, r \gg \theta.$$

The family

$$\mathbf{B} = \{B(x, r) : x \in X, r \gg \theta\}$$

is a base of a topology on X that we call topology induced by the cone metric.

Let $\varepsilon > 0$. If we are interested in obtain $c \gg \theta$ satisfying $||c|| < \varepsilon$. Then:

Since $int(P) \neq \emptyset$ we can take $e \gg \theta$ with $e \neq \theta$, and we also can choose $\lambda > 0$ such that $\lambda||e|| < \varepsilon$. If we call $c = \lambda e$, clearly $c \gg \theta$ and $||c|| = \lambda||e|| < \varepsilon$.

In the case that P is a normal cone satisfying

$$x \lesssim y \text{ implies } ||x|| \leq M \cdot ||y|| \tag{1}$$

$M \geq 1$, and if one is interested in obtaining $c \gg \theta$ with $M \cdot ||c|| < \varepsilon$, then, attending to the last paragraph it is enough to choose $\lambda > 0$ such that $\frac{\lambda}{M}||e|| < \varepsilon$.

In the next lemma we have compiled some results used in [3, 6, 7, 9], for obtaining the proofs of their results.

Lemma

Let P be a cone of E. Then:

(i) $int(P) + int(P) \subset int(P)$, [7].

(ii) $\lambda \cdot int(P) \subset int(P)$, [7].

(iii) For each $c \gg \theta$ there exists $\delta > 0$ such that $(c - x) \in int(P)$ for each $x \in X$ satisfying $||x|| < \delta$ (i.e. $x \ll c$ whenever $||x|| < \delta$, $x \in E$), [3, 9].

(iv) For each $c_1, c_2 \gg \theta$ there exists $\theta \ll c$ such that $c \ll c_1$ and $c \ll c_2$, [9].

In [3] the fixed point theorems are stated for normal cones. For instance the Banach Contraction Principle is written as follows.

Theorem

Let (X, d, E, P) be a complete cone metric space, where P is a normal cone with normal constant $M \geq 1$. Suppose that the mapping $T : X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \lesssim K \cdot d(x, y), \text{ for each } x, y \in X,$$

where $K \in]0, 1[$. Then, T has a unique fixed point in X and for each $x \in X$ the iterative sequence $\{T^n x\}$ converges to the fixed point.

This theorem is a generalization of the Banach Contraction Principle since every metric space is a cone metric space, when considering the cone $P = [0, \infty[$ of $(\mathbb{R}, |\cdot|)$.

For proving the metrizability of cone metric spaces the authors proved (Lemma 3.1, 3.2, 3.3 of [5]):

Lemma

Let (X, d) be a cone metric space. Then:

(i) For $x \in P$ and $y \in int(P)$ there exists $n \in \mathbb{N}$ such that $x \ll ny$.

(ii) If $y \in int(P)$ then $x \gtrsim y$ implies $x \in int(P)$.

(iii) $x \lesssim y \ll z$ implies $x \ll z$.

M. Asadi, S. M. Vaezpour, B. E. Rhoades, H. Soleimani, *Metrizability of cone metric spaces via renorming the Banach space*, Journal of Nonlinear Analysis and Application, **2012** (2012), Article ID jnaa-00160, doi: 10.5899/2012/jnaa-00160

Theorem

Let $(E, ||\cdot||)$ be a real Banach space with a positive cone P. Then there exists an equivalent norm on E such that P is a normal cone with normal constant $K = 1$ with respect to this norm.

They defined for each $x \in E$,

$$|||x||| = \inf\{||u|| : x \lesssim u\} + \inf\{||v|| : v \lesssim x\} + ||x||,$$

and claim that $|||\cdot|||$ is an equivalent norm to $||\cdot||$ with normal constant 1.

K. P. R. Sastry, Ch Srinivasa Rao, A. Chandra Sekhar, M. Balaiah, *On non metrizability of cone metric spaces*, International Journal of Science and Applications, **1 (3)** (2011) 1533-1535.

Example

Let $E = C^2_{\mathbb{R}}([0,1])$ with the norm

$$||f|| = ||f||_\infty + ||f'||_\infty,$$

and consider the cone $P = \{f \in E : f \geq 0\}$.

Z. Ercan, *On the end of cone metric spaces*, Topology and its Applications **166**(2014)1014.

Theorem

(Z. Ercan [2, Theorem 1.4]). Let X be a non-empty set, E be an ordered vector space with cone P and $e \in int(P)$. For a function $d : X \times X \to P$, define $\bar{d} : X \times X \to \mathbb{R}^+$ by

$$\bar{d}(x, y) = \inf\{\lambda \in \mathbb{R}^+ : d(x, y) \leq \lambda e\}.$$

(i) If d is a cone metric, then $\bar{d}$ is a metric.

(ii) If $\rho : X \times X \to \mathbb{R}^+$ is a metric, then there exists a cone metric $p : X \times X \to P$ such that $\rho = \bar{p}$.

M. A. Khamsi, *Remarks on cone metric spaces and fixed point theorems of contractive mappings*, Fixed Point Theory and Applications **2010** Article ID 315398, 7 pages, doi:10.1155/2010/315398

Theorem

Let (X, d, E, P) a cone metric space, where P is a normal cone with the normal constant $K \geq 1$. The mapping $D : X \times X \to [0, \infty[$ defined by $D(x, y) = ||d(x, y)||$ satisfies the following properties:

1. $D(x, y) = 0$ if and only if $x = y$;
2. $D(x, y) = D(y, x)$, for any $x, y \in X$;
3. $D(x, y) \leq K\left(D(x, z_1) + D(z_1, z_2) + \cdots + D(z_n, y)\right)$, for any points $x, y, z_i \in X, i = 1, \ldots, n$.

- M. Asadi, S. M. Vaezpour, B. E. Rhoades, H. Soleimani, *Metrizability of cone metric spaces via renorming teh BAnach space*, Journal of Nonlinear Analysis and Application, **2012** (2012), Article ID jnaa-00160, doi: 10.5899/2012/jnaa-00160
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D. Turkoglu, M. Abuloha, *Cone metric spaces and fixed point theorems in diametrically contractive mappings*, Acta Mathematica Sinica (2010) **26 (3)** 489-496.