Loewner's Theory: a parametric method to tackle problems in Complex Analysis

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Schedule of the lectures.

- The Bieberbach Conjecture
  1. The area theorem and the case $n = 2$ in Bieberbach Conjecture
  2. Some applications: distortion theorems
- Radial Loewner Theory
  1. First steps in Loewner theory
  2. From Loewner chains to Loewner PDE and the case $n = 3$ in Bieberbach conjecture
  3. From Loewner PDE to Loewner chains: evolution families
  4. Some applications: Univalence criteria
- Chordal Loewner Theory
- Semigroups of analytic functions
- General Loewner Theory
- Some open problems

Statement of the Bieberbach conjecture

Class $\mathcal{S}$

By $\mathcal{S}$ we denote the class of all univalent (i.e. holomorphic and injective) functions $f : \mathbb{D} \to \mathbb{C}$ normalized by

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n \qquad \text{for all } z \in \mathbb{D}.$$

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Bieberbach's Conject. (1916)

For any $f \in \mathcal{S}$,

$$|a_n| \leq n \qquad \text{for all } n.$$

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Example (The Koebe function)

Take $f : \mathbb{D} \to \mathbb{C}$ given by $f(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n$.

Then $f \in \mathcal{S}$, $f(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -1/4]$.

In particular, f gives exactly the bounds for the Bieberbach conjecture.

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In this lecture we will prove de Branges' Theorem for the easiest cases: $n = 2, 3$.

$n = 2$: It is an easy consequence of the Area Theorem.

$n = 3$: We introduce Loewner theory to solve this case.

The area theorem and the case $n = 2$

Class $\mathcal{U}$

$\mathbb{D}^* := \mathbb{D} \setminus \{0\}$. By $\mathcal{U}$ we denote the class of univalent functions $g : \mathbb{D}^* \to \mathbb{C}$ with a simple pole at zero and normalized by

$$g(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \ldots \qquad \text{for all } z \in \mathbb{D}^*.$$

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Theorem (Area Theorem)

For all $g \in \mathcal{U}$,

$$\operatorname{area}(E) = \pi \left[1 - \sum_{m=1}^{\infty} m|b_m|^2 \right],$$

where $E = \mathbb{C} \setminus g(\mathbb{D}^*)$. In particular,

$$\sum_{m=1}^{\infty} m|b_m|^2 \leq 1.$$

The area theorem and the case $n = 2$

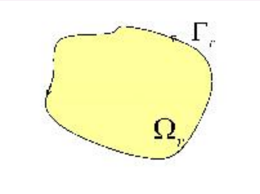
Theorem (Area Theorem)

For all $g \in \mathcal{U}$,

$$\sum_{m=1}^{\infty} m|b_m|^2 \leq 1.$$

Proof.

Fix $0 < r < 1$ and write $\Gamma_r := g(C(0,r))$.
g univalent $\Rightarrow \Gamma_r$ is a smooth Jordan curve.
Ω_r = bounded connected comp. of $\mathbb{C} \setminus \Gamma_r$.



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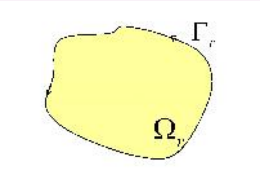

We will prove that

$$\mathrm{area}(\Omega_r) = \pi \left(\frac{1}{r^2} - \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}\right) \geq 0 \text{ for all } r < 1.$$

The area theorem and the case $n = 2$

Theorem (Area Theorem)

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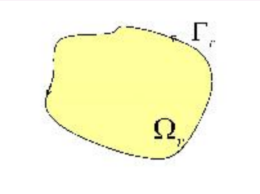
$\Omega_r =$ bounded connected comp. of $\mathbb{C} \setminus \Gamma_r$.
We parametrize Γ_r by $\gamma_r : [0, 2\pi] \to \mathbb{C}$ with

$$\gamma_r(\theta) := g(re^{-i\theta}).$$

We claim that γ_r induces a positive orientation of Γ_r (see the notes for a detailed proof).



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The area theorem and the case $n = 2$

We begin by applying Green Theorem to obtain the area of Ω_r:

$$\mathrm{area}(\Omega_r) = \int\int_{\Omega_r} 1 dxdy = \frac{1}{2}\int_{\Gamma_r^+} xdy - ydx.$$

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Since the parametrization $u(\theta) + iv(\theta) := g(re^{-i\theta})$ induces a positive orientation, we have

$$\mathrm{area}(\Omega_r) = \frac{1}{2}\int_0^{2\pi} (u(\theta)v'(\theta) - v(\theta)u'(\theta))d\theta.$$

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But

$$u(\theta)v'(\theta) - v(\theta)u'(\theta) =$$

$$= \mathrm{Re}\, g(re^{-i\theta})\mathrm{Im}\,(-ire^{-i\theta}g'(re^{-i\theta})) - \mathrm{Im}\, g(re^{-i\theta})\mathrm{Re}\,(-ire^{-i\theta}g'(re^{-i\theta}))$$

$$= -\mathrm{Re}\, g(re^{-i\theta})\mathrm{Re}\,(re^{-i\theta}g'(re^{-i\theta})) - \mathrm{Im}\, g(re^{-i\theta})\mathrm{Im}\,(re^{-i\theta}g'(re^{-i\theta}))$$

$$= -\mathrm{Re}\left[g(re^{-i\theta})\overline{re^{-i\theta}g'(re^{-i\theta})}\right].$$

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But $u(\theta)v'(\theta) - v(\theta)u'(\theta) = -\mathrm{Re}\left[g(re^{-i\theta})\overline{re^{-i\theta}g'(re^{-i\theta})}\right]$.

Thus

$$\begin{aligned}\mathrm{area}(\Omega_r) &= -\frac{1}{2}\int_0^{2\pi} \mathrm{Re}\left[g(re^{-i\theta})\overline{re^{-i\theta}g'(re^{-i\theta})}\right] d\theta \\ &= -\frac{1}{2}\mathrm{Re}\int_0^{2\pi} g(re^{-i\theta})\overline{re^{-i\theta}g'(re^{-i\theta})}d\theta.\end{aligned}$$

The area theorem and the case $n = 2$

Let us calculate the integrand

$$g(re^{-i\theta})\overline{re^{-i\theta}g'(re^{-i\theta})} =$$

$$= \left(\frac{1}{r}e^{i\theta} + \sum_{n=0}^{\infty} b_n r^n e^{-in\theta}\right) \overline{re^{-i\theta}\left(-\frac{1}{r^2}e^{2i\theta} + \sum_{n=1}^{\infty} nb_n r^{n-1} e^{-i(n-1)\theta}\right)}$$

$$= \left(\frac{1}{r}e^{i\theta} + \sum_{n=0}^{\infty} b_n r^n e^{-in\theta}\right)\left(-\frac{1}{r}e^{-i\theta} + \sum_{n=1}^{\infty} n\overline{b_n} r^n e^{in\theta}\right)$$

$$= -\frac{1}{r^2} + \sum_{n=1}^{\infty} n\overline{b_n} r^{n-1} e^{i(n+1)\theta} - \sum_{n=0}^{\infty} b_n r^{n-1} e^{-i(n+1)\theta} +$$

$$+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} b_n m\overline{b_m} r^{r+m} e^{i(m-n)\theta}.$$

The area theorem and the case $n = 2$

$$g(re^{-i\theta})\overline{re^{-i\theta}g'(re^{-i\theta})} = -\frac{1}{r^2} + \sum_{n=1}^{\infty} n\overline{b_n}r^{n-1}e^{i(n+1)\theta} - \sum_{n=0}^{\infty} b_n r^{n-1}e^{-i(n+1)\theta} + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} b_n m\overline{b_m} r^{r+m} e^{i(m-n)\theta}.$$

Since $\int_0^{2\pi} e^{ik\theta}d\theta = 0$ for $k \neq 0$ and $\int_0^{2\pi} e^{i0\theta}d\theta = 2\pi$ we have

$$\begin{aligned} 0 \leq \mathrm{area}(\Omega_r) &= -\frac{1}{2}\mathrm{Re}\int_0^{2\pi} g(re^{-i\theta})\overline{re^{-i\theta}g'(re^{-i\theta})}d\theta \\ &= -\frac{1}{2}\left(-\frac{1}{r^2}2\pi + \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} 2\pi\right) \\ &= \pi\left(\frac{1}{r^2} - \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}\right). \end{aligned}$$

We let r go to 1 and we obtain the result.

The area theorem and the case $n = 2$

Theorem (Area Theorem)

For all $g \in \mathcal{U}$,

$$\sum_{m=1}^{\infty} m|b_m|^2 \leq 1. \tag{2.1}$$

In particular,

Corollary

For all $g \in \mathcal{U}$, we have

1. $|b_1| \leq 1$;
2. $|b_1| = 1$ *if and only if* $g(z) = \frac{1}{z} + b_0 + \lambda z$ *for some* $\lambda \in \partial\mathbb{D}$.

The area theorem and the case $n = 2$

Take now $f \in \mathcal{S}$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.
Then

$$\frac{1}{f(z)} = \frac{1}{z} - a_2 + (a_2^2 - a_3)z + ... \in \mathcal{U}$$

Therefore $|a_2^2 - a_3| \leq 1$. But this is not enough. We have to work a little bit more.

The area theorem and the case $n = 2$

Corollary

For all $f \in \mathcal{S}$, we have

1. $|a_2| \leq 2$;
2. $|a_2| = 2$ *if and only if* $f(z) = \frac{z}{(1-\lambda z)^2}$ *for some* $\lambda \in \partial\mathbb{D}$.

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For all $f \in \mathcal{S}$, we have

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Proof. Write $\varphi(z) = \frac{f(z)}{z}$. Defining $\varphi(0) = 1$, φ is analytic in $\mathbb{D}$.

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$f(0) = 0$ + univalence $\Rightarrow \varphi$ has no zero:
there is $h \in \mathcal{H}(\mathbb{D})$ with $h(0) = 1$ and $h(z)^2 = \varphi(z)$.
Write $k(z) = zh(z^2)$. Then

$$k(z)^2 = z^2 h(z^2)^2 = z^2 \varphi(z^2) = f(z^2).$$

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k is univalent because if $k(z) = k(w)$, then $f(z^2) = f(w^2)$. That is, $z^2 = w^2$. If $z = -w$, then $k(z) = k(w) = k(-z) = -k(z)$. Thus $k(z) = 0$ and $z = w = 0$. That is, $z = w$ and k is univalent.

The area theorem and the case $n = 2$

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k is univalent
Some easy computations show that $k(z) = z + \frac{1}{2}a_2 z^3 + ...$
Take $G(z) = \frac{1}{k(z)} \in \mathcal{U}$ and $G(z) = \frac{1}{z} - \frac{1}{2}a_2 z + ...$
By above corollary, $|\frac{1}{2}a_2| \leq 1$. This proves (1).

The area theorem and the case $n = 2$

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Proof.

Recall that

$$\varphi(z) = \frac{f(z)}{z}, \; h(z)^2 = \varphi(z), \; k(z) = zh(z^2), \text{ and } G(z) = \frac{1}{k(z)}.$$

The area theorem and the case $n = 2$

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Proof. $|a_2| = 2$ if and only if $G(z) = \frac{1}{z} - \lambda z$ for some $\lambda \in \partial\mathbb{D}$.

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Thus $\varphi(z) = \frac{1}{(1-\lambda z)^2}$ and $f(z) = \frac{z}{(1-\lambda z)^2}$.

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The area theorem and the case $n = 2$

An straightforward consequence of the above corollary is:

Theorem (Koebe's $\frac{1}{4}$-Theorem)

If $f \in \mathcal{S}$, then $D(0, 1/4) \subset f(\mathbb{D})$.

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Proof. Take $w_0 \notin f(\mathbb{D})$ and define $g : \mathbb{D} \to \mathbb{C}$ as $g(z) = \frac{w_0 f(z)}{w_0 - f(z)}$.

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Since g is the composition of f with a Möbius transformation, it is univalent.
Moreover, $g(z) = z + \left(a_2 + \frac{1}{w_0}\right) z^2 + \ldots \in \mathcal{S}$.

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By above corollary, applied to f and g, we have

$$\left| a_2 + \frac{1}{w_0} \right| \leq 2, \qquad |a_2| \leq 2.$$

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By above corollary, applied to f and g, we have

$$\left| a_2 + \frac{1}{w_0} \right| \leq 2, \qquad |a_2| \leq 2.$$

Hence $\left|\frac{1}{w_0}\right| \leq \left| a_2 + \frac{1}{w_0} \right| + |a_2| \leq 4. \Rightarrow \frac{1}{4} \leq |w_0|$.

Classical distortion theorems

We obtain a group of inequalities that constitute, jointly with the Area Theorem, the first sharp results for univalent functions. They are equiv. to $|a_2| \leq 2$ and are needed to prove $|a_3| \leq 3$.

Classical distortion theorems

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Theorem (Koebe distortion Theorem or Growth Theorem)

If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2},$$

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3},$$

In each case, equality holds if and only if f is a suitable rotation of the Koebe function.

Classical distortion theorems

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Lemma

If $f \in \mathcal{S}$, then

$$\left| z\frac{f''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2} \right| \leq \frac{4|z|}{1-|z|^2} \quad (z \in \mathbb{D}).$$

In particular,

$$\frac{2|z|^2 - 4|z|}{1-|z|^2} \leq \operatorname{Re}\left[z\frac{f''(z)}{f'(z)}\right] \leq \frac{2|z|^2 + 4|z|}{1-|z|^2} \quad (z \in \mathbb{D}).$$

Classical distortion theorems

Theorem (Koebe distortion Theorem)

If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

Proof. The function f' is analytic and nonvanishing on the unit disk, therefore we may choose an analytic branch of its logarithm. Write $z = re^{i\theta}$. Since

$$\begin{aligned}\frac{\partial}{\partial r}\log|f'(re^{i\theta})| &= \frac{\partial}{\partial r}\mathrm{Re}\left[\log f'(re^{i\theta})\right] = \mathrm{Re}\left[\frac{\partial}{\partial r}\log f'(re^{i\theta})\right] \\ &= \mathrm{Re}\left[\frac{f''(re^{i\theta})}{f'(re^{i\theta})}e^{i\theta}\right] = \frac{1}{r}\mathrm{Re}\left[\frac{f''(re^{i\theta})}{f'(re^{i\theta})}re^{i\theta}\right].\end{aligned}$$

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Classical distortion theorems

Theorem (Koebe distortion Theorem)

If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then

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Proof. ... Since $\frac{\partial}{\partial r} \log|f'(re^{i\theta})| = \frac{1}{r}\mathrm{Re}\left[\frac{f''\left(re^{i\theta}\right)}{f'\left(re^{i\theta}\right)} re^{i\theta}\right]$ and

$$\frac{2|z|^2-4|z|}{1-|z|^2} \leq \mathrm{Re}\left[z\frac{f''(z)}{f'(z)}\right] \leq \frac{2|z|^2+4|z|}{1-|z|^2} \quad (z \in \mathbb{D})$$

we obtain

$$\frac{2r-4}{1-r^2} \leq \frac{\partial}{\partial r} \log|f'(re^{i\theta})| \leq \frac{2r+4}{1-r^2}.$$

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Classical distortion theorems

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Proof. ...

$$\frac{2r-4}{1-r^2} \leq \frac{\partial}{\partial r} \log |f'(re^{i\theta})| \leq \frac{2r+4}{1-r^2}.$$

Now, we integrate in r to obtain

$$\log \frac{1-r}{(1+r)^3} \leq \log |f'(re^{i\theta})| \leq \log \frac{1+r}{(1-r)^3},$$

where we have used that $\log f'(0) = 0$. We get the inequalities by taking exponentiation.

Classical distortion theorems

Theorem (Koebe distortion Theorem)

If $f \in \mathcal{S}$ and $z \in \mathbb{D}$, then

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Proof. We may assume that $z \neq 0$. If $z = re^{i\theta}$, then

$$f(z) = \int_0^r f'(\rho e^{i\theta}) e^{i\theta} d\rho$$

and, by the previous bound for the derivative, we have

$$|f(z)| \leq \int_0^r |f'(\rho e^{i\theta})| d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)^3} d\rho = \frac{r}{(1-r)^2}.$$

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First steps in Loewner theory

The Loewner's idea was to introduce a parameter in Taylor coefficients of a univalent function without losing the univalence and with some additional properties in order to be able to differentiate with respect to the parameter and take advantage of such derivation.

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Loewner worked with Riemann maps of slit domains (the complex plane minus a Jordan arc).
For some applications this is not a real restriction because such family of functions is dense in $\mathcal{S}$ in the topology of uniform convergence on compacta.
Some years later, Kufarev and Pommerenke generalized such idea to general univalent functions.
We adopt this point of view.

First steps in Loewner theory

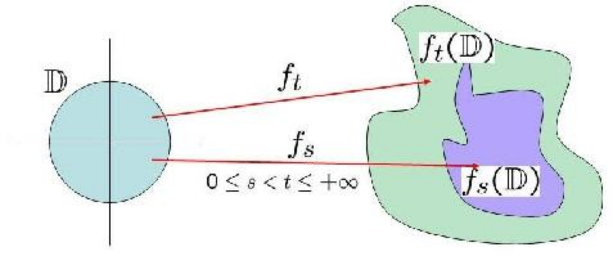
Definition

A family of holomorphic functions in $\mathbb{D}$, $(f_t)_{t\geq 0}$, is said to be a *(radial) Loewner chain*, if

LC1 all f_t's are univalent in $\mathbb{D}$;

LC2 $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s \leq t$;

LC3 for any $t \geq 0$, $f_t(z) = e^t z + a_2(t)z^2 + \ldots$, i.e., $e^{-t}f_t \in \mathcal{S}$.



First steps in Loewner theory

Since $e^{-t} f_t \in \mathcal{S}$, the Koebe distortion theorem shows that

$$e^t \frac{|z|}{(1+|z|)^2} \leq |f_t(z)| \leq e^t \frac{|z|}{(1-|z|)^2},$$

$$e^t \frac{1-|z|}{(1+|z|)^3} \leq |f_t'(z)| \leq e^t \frac{1+|z|}{(1-|z|)^3}.$$

Moreover, by Koebe 1/4-theorem, $e^t D(0, 1/4) \subset f_t(\mathbb{D})$. In particular,

$$\bigcup_{t \geq 0} f_t(\mathbb{D}) = \mathbb{C}.$$

First steps in Loewner theory

Since f_t are univalent and $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s \leq t$, we can define the analytic self-maps of the unit disk $\varphi_{s,t} := f_t^{-1} \circ f_s$ for all $0 \leq s \leq t < +\infty$.

Clearly, $\varphi_{s,t}$ is univalent, $\varphi_{s,t}(0) = 0$, and $\varphi'_{s,t}(0) = e^{s-t}$.

There are different names in the literature for this biparametric family $(\varphi_{s,t})$.
We call them *evolution family* associated with the Loewner chain (f_t).

First steps in Loewner theory

Lemma

If (f_t) *is a Loewner chain with evolution family* $(\varphi_{s,t})$ *then, for* $0 \leq s \leq t \leq u < +\infty$,

$$|f_t(z) - f_s(z)| \leq \frac{8|z|}{(1-|z|)^4}(e^t - e^s) \quad (z \in \mathbb{D}),$$

$$|\varphi_{t,u}(z) - \varphi_{s,u}(z)| \leq \frac{2|z|}{(1-|z|)^2}(1 - e^{s-t}) \quad (z \in \mathbb{D}).$$

First steps in Loewner theory

Let $p : \mathbb{D} \to \mathbb{C}$ be analytic with $\operatorname{Re} p(z) \geq 0$ for all z. Then there exists an increasing function $\mu : [0, 2\pi] \to [0, +\infty)$ such that $\mu(2\pi) - \mu(0) = \operatorname{Re} p(0)$ and

$$p(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) + i \operatorname{Im} p(0).$$

Therefore

Corollary

Let $p : \mathbb{D} \to \mathbb{C}$ be analytic with $\operatorname{Re} p(z) \geq 0$ for all $z \in \mathbb{D}$ and $p(0) = 1$. Then $|p(z)| \leq \frac{1+|z|}{1-|z|}$ for all $z \in \mathbb{D}$.

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Proof. In our case, $\operatorname{Im} p(0) = 0$:

$$|p(z)| \leq \int_0^{2\pi} \left| \frac{e^{it} + z}{e^{it} - z} \right| d\mu(t) \leq \int_0^{2\pi} \frac{1 + |z|}{1 - |z|} d\mu(t) = \frac{1 + |z|}{1 - |z|}.$$

First steps in Loewner theory

$$|f_t(z) - f_s(z)| \leq \frac{8|z|}{(1-|z|)^4}(e^t - e^s) \quad (z \in \mathbb{D}).$$

Proof.

$|a| \leq |b|$, then Re $\frac{b-a}{b+a} = \frac{|b|^2-|a|^2}{|b+a|^2} \geq 0$.

First steps in Loewner theory

$$|f_t(z) - f_s(z)| \le \frac{8|z|}{(1-|z|)^4}(e^t - e^s) \quad (z \in \mathbb{D}).$$

Proof. Since $|\varphi_{s,t}(z)| < |z| < 1$ for $s < t$ we see that

$$p(z,s,t) = \frac{1+e^{s-t}}{1-e^{s-t}} \frac{z-\varphi_{s,t}(z)}{z+\varphi_{s,t}(z)}$$

has positive real part in $\mathbb{D}$.
(because if $|a| \le |b|$, then Re $\frac{b-a}{b+a} = \frac{|b|^2-|a|^2}{|b+a|^2} \ge 0$.)

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By the previous corollary

$$|p(z,s,t)| \leq \frac{1+|z|}{1-|z|}.$$

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First steps in Loewner theory

$$|f_t(z) - f_s(z)| \leq \frac{8|z|}{(1-|z|)^4}(e^t - e^s) \quad (z \in \mathbb{D}).$$

Proof.

$$|p(z,s,t)| \leq \frac{1+|z|}{1-|z|} \quad \Rightarrow \quad \left|\frac{z-\varphi_{s,t}(z)}{z+\varphi_{s,t}(z)}\right| \leq \frac{1+|z|}{1-|z|}\frac{1-e^{s-t}}{1+e^{s-t}}$$

and

$$|z-\varphi_{s,t}(z)| \leq 2|z|\frac{1+|z|}{1-|z|}\frac{1-e^{s-t}}{1+e^{s-t}}.$$

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Thus

$$\begin{aligned} |f_t(z) - f_s(z)| &= |f_t(z) - f_t(\varphi_{s,t}(z))| = \left|\int_{\varphi_{s,t}(z)}^{z} f_t'(\xi)d\xi\right| \\ &\le |z-\varphi_{s,t}(z)|\frac{2e^t}{(1-|z|)^3} \le \frac{8|z|}{(1-|z|)^4}(e^t - e^s). \end{aligned}$$

First steps in Loewner theory

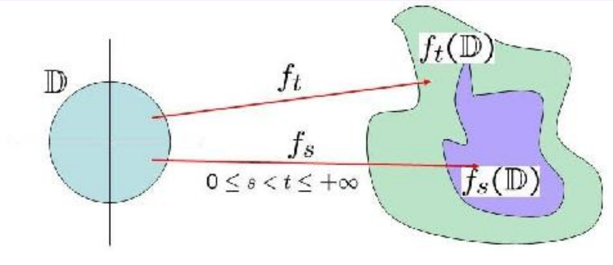
With a similar argument we have

$$|\varphi_{t,u}(z) - \varphi_{s,u}(z)| \leq \frac{2|z|}{(1-|z|)^2}(1 - e^{s-t}) \quad (z \in \mathbb{D}).$$

First steps in Loewner theory

Theorem

For any $f \in \mathcal{S}$, there exists a Loewner chain (f_t) such that $f = f_0$.



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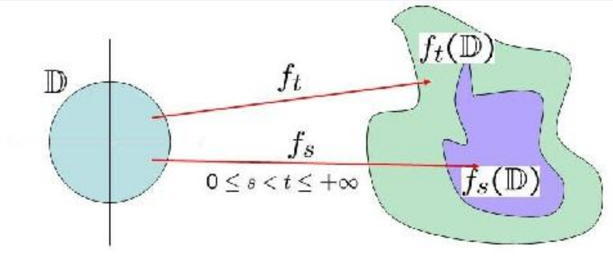

To prove this result we need Carathéodory kernel theorem.

Carathéodory kernel theorem

Carathéodory gave a complete geometric characterization of the converge of univalent functions in terms of the convergence of their image domains.

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Definition

Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of open domains in $\mathbb{C}$, with $0 \in \Omega_n$ for every $n \in \mathbb{N}$. We define the kernel of $\{\Omega_n\}$ to be the set Ω as follows:

1. if 0 is not an interior point of $\cap_{n\in\mathbb{N}}\Omega_n$, then $\Omega := \{0\}$;
2. if 0 is an interior point of $\cap_{n\in\mathbb{N}}\Omega_n$, then Ω is defined as the set of all points $w \in \mathbb{C}$ such that there exists a domain H with $0 \in H$, $w \in H$ such that $H \subset \Omega_n$ for all suffic. large n.

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We say that the sequence of domains $\{\Omega_n\}_{n\in\mathbb{N}}$ converges to Ω if each subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ has kernel Ω. In such a case, we write $\Omega_n \to \Omega$.

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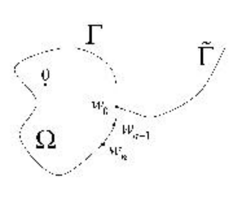
Example

If $\Omega_{n-1} \subset \Omega_n$ for any n, then the kernel is $\Omega = \cup_n\Omega_n$.

Carathéodory kernel theorem

Example.
Let Γ be a Jordan curve, take Ω the unique bounded connected component of $\mathbb{C} \setminus \Gamma$. Assume that $0 \in \Omega$.



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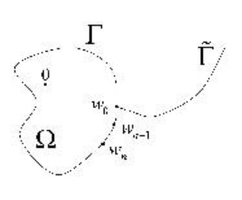

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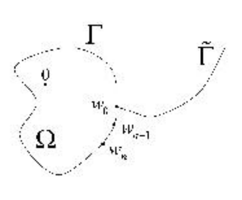
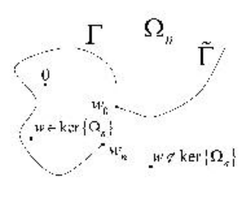
Let $\{w_n\} \subset \Gamma$ such that w_n moves clockwise on Γ and such that $w_n \to w_0$. Consider $\gamma_n := \tilde{\Gamma} \cup$ " the portion of Γ from w_0 to w_n".

and $\Omega_n = \mathbb{C} \setminus \gamma_n$.





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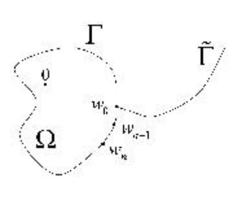

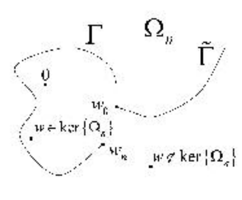

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Theorem (Carathéodory kernel theorem)

Let $\{\Omega_n\}_{n\in\mathbb{N}}$ *be a sequence of open domains in* $\mathbb{C}$, *with* $0 \in \Omega_n$ *for every* $n \in \mathbb{N}$.

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Furthermore, $f(\mathbb{D}) = \Omega$.

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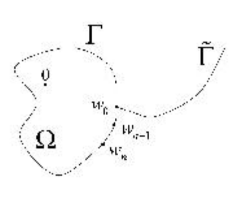

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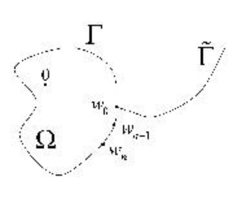

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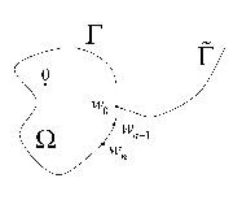
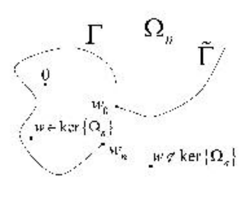
and $\Omega_n = \mathbb{C} \setminus \gamma_n$.





Carathéodory kernel theorem

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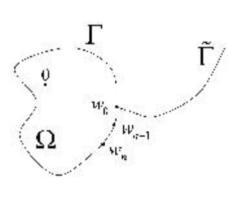

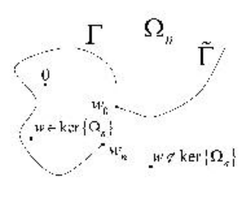

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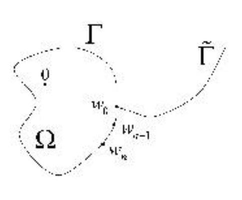

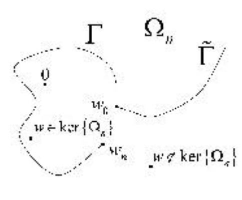

Therefore the Riemann map of Ω can be approximated by the Riemann map of a family of slit domains.

From this fact, one can prove that $\mathcal{S}'$ is dense in $\mathcal{S}$ where

$\mathcal{S}' = \{f \in \mathcal{S} : f(\mathbb{D})$ is the whole plane $\mathbb{C}$ slit along a Jordan arc $\Gamma\}$.

First steps in Loewner theory

Theorem

For any $f \in \mathcal{S}$, there exists a Loewner chain (f_t) such that $f = f_0$.

Proof. First assume that f is univalent in a neighborhood of $\overline{\mathbb{D}}$.
Thus $\gamma = f(\partial\mathbb{D})$ is a Jordan curve.
Notation: $\widehat{\mathbb{C}}$ the Riemann sphere;
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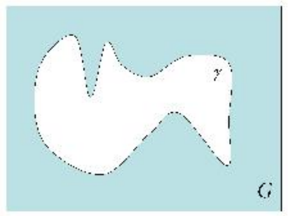

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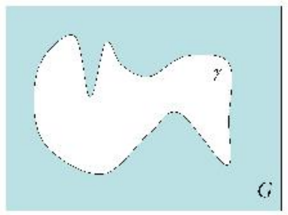
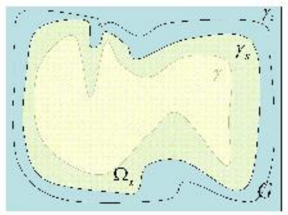
Take $g : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to G$ univalent and onto such that $g(\infty) = \infty$.
For $0 \leq t < +\infty$, denote by Ω_t the inner domain of the Jordan curve
$\gamma_t := \{g(\xi) : |\xi| = e^t\}$.





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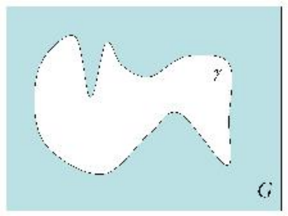
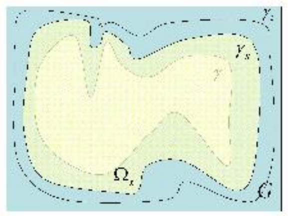
For $0 \leq t < +\infty$, denote by Ω_t the inner domain of the Jordan curve $\gamma_t := \{g(\xi) : |\xi| = e^t\}$. Notice that so far we have built a family of domains Ω_t such that $\Omega_0 = f(\mathbb{D})$, $\bigcup_{t \geq 0} \Omega_t = \mathbb{C}$, and $\Omega_s \subseteq \Omega_t$ for all $s < t$.





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For any $f \in \mathcal{S}$, there exists a Loewner chain (f_t) such that $f = f_0$.

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$$g_t(0) = 0 \text{ and } \beta(t) = g_t'(0) > 0.$$

Schawrz's lemma implies that $\beta(s) < \beta(t)$ because $g_t^{-1} \circ g_s$ is a self-map of the unit disk s,t,

$$g_t^{-1} \circ g_s(0) = 0 \quad \text{and} \quad (g_t^{-1} \circ g_s)'(0) = \beta(s)/\beta(t).$$

By uniqueness of such Riemann map $g_0 = f$ (because $\gamma = \gamma_0$).

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(g_t) is not the family we are looking for because we do not know if $g_t'(0) = \beta(t) = e^t$.

First steps in Loewner theory

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For any $f \in \mathcal{S}$, there exists a Loewner chain (f_t) such that $f = f_0$.

Proof.

Since the family of domains increase, $\Omega_{t_n} \to \Omega_{t_0}$ if $t_n \to t_0 < +\infty$ and $\Omega_{t_n} \to \mathbb{C}$ if $t_n \to +\infty$.

The Carathéodory kernel theorem shows g_t goes to g_s, and then g_t' goes to g_s', as t tends s. This implies that the function $\beta(t) = g_t'(0)$ is continuous in $[0, +\infty)$ and $\beta(t) \to +\infty$ as $t \to \infty$.

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The proof is finished in case f in univalent beyond the unit disk.

First steps in Loewner theory

Theorem

For any $f \in \mathcal{S}$, there exists a Loewner chain (f_t) such that $f = f_0$.

Proof. For the general case, let f be an arbitrary function in $\mathcal{S}$. For each n, put $r_n = 1 - 1/n$ and let $f_n(z) := \quad f(r_n z)$.

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Every sequence of Loewner chains (f_t^n) has a subsequence that converges to a Loewner chain (f_t) locally uniformly in $\mathbb{D}$ for each $t \geq 0$.

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Lemma

Every sequence of Loewner chains (f_t^n) has a subsequence that converges to a Loewner chain (f_t) locally uniformly in $\mathbb{D}$ for each $t \geq 0$.

Thus, some subsequence of (f_t^n) converges to a Loewner chain (f_t). It is routine to check that $f_0 = f$.

From Loewner chains to Loewner PDE

Theorem

Let (f_t) be a Loewner chain. Then there exists a function $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ such that

1. $z \mapsto p(z, t)$ *is analytic for all* t;
2. $t \mapsto p(z, t)$ *is measurable for all* z;
3. $p(0, t) = 1$ *for all* t;
4. $\operatorname{Re} p(z, t) > 0$ *for all* z *and all* t;

and, for almost all t,

$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) p(z, t). \qquad \text{(Loewner PDE)}$$

Definition

A function $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ satisfying (1), (2), (3) and (4) is called a Herglotz function.

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What can we do? There is a null-set E such that $\frac{\partial f_t(1/k)}{\partial t}$ exists for all $t \in [0, +\infty) \setminus E$ and all $k \in \mathbb{N}$.

...

From Loewner chains to Loewner PDE

Theorem (Vitali's Theorem)

Let Ω be a domain in $\mathbb{C}$ and $\{g_n\}$ a sequence in $\mathrm{Hol}(\Omega)$. Assume that for every compact set K of Ω there exists a constant $M > 0$ such that

$$|g_n(z)| \leq M \quad \text{for all } z \in K \text{ and all } n.$$

Assume that the set

$$A := \left\{ z \in \Omega : \lim_n g_n(z) \text{ exists} \right\}$$

has an accumulation point in Ω.
Then there is $g \in \mathrm{Hol}(\Omega)$ such that $\lim_n g_n = g$ uniformly on compact subsets of Ω.

From Loewner chains to Loewner PDE

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Let (f_t) be a Loewner chain. Then there exists a Herglotz function p such that $\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z,t)$.

Proof. ... Fix $t \in [0, +\infty) \setminus E$ and any sequence $t_n \in [0, +\infty)$ converging to t, with $t_n \neq t$.

Take $(g_n) = \left(\frac{f_{t_n} - f_t}{t_n - t}\right)$.

Since $|f_t(z) - f_s(z)| \leq \frac{8|z|}{(1-|z|)^4}(e^t - e^s)$, the sequence (g_n) is bounded on compact sets.

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Moreover, the set

$$A = \left\{ z \in \mathbb{D} : \lim_n \frac{f_{t_n}(z) - f_t(z)}{t_n - t} \text{ exists} \right\}$$

contains the points $1/k$ for all $k \in \mathbb{N}$ and, therefore, has an accumulation point in $\mathbb{D}$. ...

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Let (f_t) be a Loewner chain. Then there exists a Herglotz function p such that $\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z,t)$.

Proof. ...

Then there is an analytic function in the unit disk h such that $\lim_n \frac{f_{t_n}-f_t}{t_n-t} = h$. The arbitrariness of the sequence t_n implies that $\lim_{s\to t} \frac{f_s-f_t}{s-t} = h$.

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Let (f_t) be a Loewner chain. Then there exists a Herglotz function p such that $\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z,t)$.

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Using that $f_s(z) = f_t(\varphi_{s,t}(z))$, we can write

$$\frac{f_t(z) - f_s(z)}{t-s} = \frac{e^{t-s}-1}{t-s}\frac{z+\varphi_{s,t}(z)}{e^{t-s}+1}\frac{f_t(z)-f_s(z)}{z-\varphi_{s,t}(z)}p(z,s,t)$$

where $p(z,s,t)$ is again defined by (7) and $\operatorname{Re} p(z,s,t) \geq 0$.

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where $p(z,s,t)$ is again defined by (7) and $\operatorname{Re} p(z,s,t) \geq 0$.
Since $f_t \to f_s$ as $t \to s$ locally uniformly in $\mathbb{D}$, we have $f_t' \to f_s'$.
Since $\varphi_{s,t}(z) \to z$ it follows that, as $t \to s$

$$\frac{f_t(z) - f_s(z)}{z - \varphi_{s,t}(z)} = \int_0^1 f_t'(z + \lambda[z - \varphi_{s,t}(z)])d\lambda \to f_s'(z).$$

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Let (f_t) be a Loewner chain. Then there exists a Herglotz function p such that $\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z,t)$.

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and bearing in mind that $\frac{f_t(z)-f_s(z)}{z-\varphi_{s,t}(z)} \to f_t'(z)$ we obtain that

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for some function $p(z,s)$ which again has non-negative real part.

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and bearing in mind that $\frac{f_t(z) - f_s(z)}{z - \varphi_{s,t}(z)} \to f_t'(z)$ we obtain that

$$\frac{\partial f_s(z)}{\partial s} = zf_s'(z)p(z, s)$$

for some function $p(z, s)$ which again has non-negative real part. Such a function is measurable in s because $p(z, s)$ is the limit of $p(z, s, t)$ and this function is continuous in s for all t.

The coefficient a_3

Theorem

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in $\mathcal{S}$. Then $|a_3| \leq 3$.

Idea behind the proof

Pass bounds about Taylor coefficients of p to Taylor coefficients of f_t using Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) p(z, t).$$

The coefficient a_3

Lemma

Let $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots$ be analytic in $\mathbb{D}$ with $\operatorname{Re} p(z) \geq 0$ for all $z \in \mathbb{D}$. Then

1. $(\operatorname{Re} c_1)^2 \leq 2 + \operatorname{Re} c_2$;
2. $|c_n| \leq 2$ *for all n.*

Proof. (1) $\psi(z) := \frac{1}{z}\frac{1-p(z)}{1+p(z)}$ is an analytic self-map of $\mathbb{D}$.
By Schwarz's Lemma we have that $|\psi'(0)| \leq 1 - |\psi(0)|^2$. That is

$$\left|\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right| \leq 1 - \frac{1}{4}|c_1|^2.$$

The coefficient a_3

Lemma

Let $p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ *be analytic in* $\mathbb{D}$ *with* $\operatorname{Re} p(z) \geq 0$ *for all* $z \in \mathbb{D}$. *Then*

1. $(\operatorname{Re} c_1)^2 \leq 2 + \operatorname{Re} c_2$;
2. $|c_n| \leq 2$ *for all* n.

Proof. (1) $\psi(z) := \frac{1}{z}\frac{1-p(z)}{1+p(z)}$ is an analytic self-map of $\mathbb{D}$.
By Schwarz's Lemma we have that $|\psi'(0)| \leq 1 - |\psi(0)|^2$. That is

$$-\operatorname{Re}\left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right) \leq \left|\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right| \leq 1 - \frac{1}{4}|c_1|^2.$$

Taking the negative real part on the left-hand side we conclude that

$$2 + \operatorname{Re} c_2 \geq \frac{1}{2}\operatorname{Re} c_1^2 + \frac{1}{2}|c_1|^2 = (\operatorname{Re} c_1)^2.$$

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1. $(\operatorname{Re} c_1)^2 \leq 2 + \operatorname{Re} c_2$;
2. $|c_n| \leq 2$ *for all n.*

Proof. (2) There is an increasing function $\mu : [0, 2\pi] \to \mathbb{R}$ such that $\mu(2\pi) - \mu(0) = 1$ and

$$p(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t).$$

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$$p(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t).$$

Since $\frac{e^{it}+z}{e^{it}-z} = 1 + 2\sum_{n=1}^{\infty} e^{-itn} z^n$, we deduce that $c_n = 2\int_0^{2\pi} e^{-itn} d\mu(t)$, so that

$$|c_n| \leq 2 \int_0^{2\pi} |e^{-itn}| d\mu(t) = 2(\mu(2\pi) - \mu(0)) = 2.$$

$|a_3| \leq 3$

Proof. Take any $\lambda \in \partial\mathbb{D}$. Then the function

$$g(z) = \overline{\lambda} f(\lambda z) = z + \lambda a_2 z^2 + \lambda^2 a_3 z^3 + \ldots$$

also belongs to $\mathcal{S}$. Since it is equivalent to prove our result for g or f, we may assume that $a_3 \geq 0$.

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also belongs to $\mathcal{S}$. Since it is equivalent to prove our result for g or f, we may assume that $a_3 \geq 0$.
Take (f_t) a Loewner chain such that $f_0 = f$.
We know that there is a Herglotz function p such that

$$\begin{aligned} & e^t z + a_2'(t) z^2 + a_3'(t) z^3 + \ldots \\ = & z(1 + c_1(t)z + c_2(t)z^2 + c_3(t)z^3 + \ldots)(e^t + 2a_2(t)z + 3a_3(t)z^2 + \ldots), \end{aligned}$$

where

$$f_t(z) = e^t z + \sum_{n=2}^{+\infty} a_n(t) z^n \qquad \text{for all } z \in \mathbb{D},$$

$$p(z,t) = 1 + c_1(t)z + c_2(t)z^2 + c_3(t)z^3 + \ldots \qquad \text{for all } z \in \mathbb{D}.$$

$|a_3| \leq 3$

Proof. Equalling coefficients of z^2 and z^3:

$a_2'(t) = 2a_2(t) + c_1(t)e^t$ and $a_3'(t) = 3a_3(t) + c_1(t)2a_2(t) + c_2(t)e^t$.

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Integrating the first equation:

$$a_2(t) = -e^{2t}\left(\int_t^\infty e^{-\xi}c_1(\xi)d\xi + C\right).$$

Since $e^{-t}f_t \in \mathcal{S}$, we have that

$$|e^{-t}a_2(t)| = e^t\left|\int_t^\infty e^{-\xi}c_1(\xi)d\xi + C\right| \leq 2.$$

This implies that $C = 0$ and

$$a_2(t) = -e^{2t}\int_t^\infty e^{-\xi}c_1(\xi)d\xi, \quad a_2 = a_2(0) = -\int_0^\infty e^{-\xi}c_1(\xi)d\xi$$

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Proof. From $a_3'(t) = 3a_3(t) + c_1(t)2a_2(t) + c_2(t)e^t$ we deduce

$$a_3(t) = -e^{3t}\left(\int_t^\infty \left(e^{-2\xi}c_2(\xi) + 2e^{-3\xi}a_2(\xi)c_1(\xi)\right)d\xi + C\right).$$

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Lemma

$|a_3 - a_2^2| \leq 1$. *In particular,* $|a_3| \leq 5$.
Hint: The function $g(z) = 1/f(z)$ *belongs to* $\mathcal{U}$ *and*

$$g(z) = \frac{1}{z} - a_2 + (a_3 - a_2)^2 z + \ldots$$

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Thus

$$|e^{-t}a_3(t)| = e^{2t}\left|\int_t^\infty \left(e^{-2\xi}c_2(\xi) + 2e^{-3\xi}a_2(\xi)c_1(\xi)\right)d\xi + C\right| \leq 5.$$

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And C=0.

$|a_3| \leq 3$

Proof.

$$
\begin{aligned}
a_3 &= a_3(0) = -\int_0^\infty \left(e^{-2\xi}c_2(\xi) + 2e^{-3\xi}a_2(\xi)c_1(\xi)\right) d\xi \\
&= -\int_0^\infty e^{-2\xi}c_2(\xi)d\xi - 2\int_0^\infty e^{-3\xi}c_1(\xi)\left[-e^{2\xi}\int_\xi^\infty e^{-s}c_1(s)ds\right] d\xi
\end{aligned}
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&\quad \left[\text{writing } k(\xi) := \int_\xi^\infty e^{-s}c_1(s)ds\right] \\
&= -\int_0^\infty e^{-2\xi}c_2(\xi)d\xi - 2\int_0^\infty k'(\xi)k(\xi)d\xi \\
&= -\int_0^\infty e^{-2\xi}c_2(\xi)d\xi + k(0)^2 \\
&= -\int_0^\infty e^{-2\xi}c_2(\xi)d\xi + \left(\int_0^\infty e^{-\xi}c_1(\xi)d\xi\right)^2 .
\end{aligned}$$

$|a_3| \leq 3$

Proof. Finally, we have

$$a_3 = \operatorname{Re} a_3 = -\int_0^{\infty} e^{-2\xi} \operatorname{Re} c_2(\xi) d\xi + \operatorname{Re} \left(\int_0^{\infty} e^{-\xi} c_1(\xi) d\xi \right)^2$$

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$$\left[\text{ using } (\operatorname{Re} c_1)^2 \leq 2 + \operatorname{Re} c_2 \text{ and } \operatorname{Re}(w^2) \leq (\operatorname{Re} w)^2\right]$$

$$\leq \int_0^\infty e^{-2\xi}(2 - (\operatorname{Re} c_1(\xi))^2)d\xi + \left(\int_0^\infty e^{-\xi} \operatorname{Re} c_1(\xi)d\xi\right)^2$$

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Proof. Finally, we have

$$
\begin{aligned}
a_3 &= \operatorname{Re} a_3 = -\int_0^\infty e^{-2\xi} \operatorname{Re} c_2(\xi)\,d\xi + \operatorname{Re}\left(\int_0^\infty e^{-\xi} c_1(\xi)\,d\xi\right)^2 \\
&\qquad \left[\text{ using } (\operatorname{Re} c_1)^2 \leq 2 + \operatorname{Re} c_2 \text{ and } \operatorname{Re}(w^2) \leq (\operatorname{Re} w)^2\right] \\
&\leq \int_0^\infty e^{-2\xi}(2 - (\operatorname{Re} c_1(\xi))^2)\,d\xi + \left(\int_0^\infty e^{-\xi} \operatorname{Re} c_1(\xi)\,d\xi\right)^2 \\
&= 1 - \int_0^\infty e^{-2\xi}(\operatorname{Re} c_1(\xi))^2\,d\xi + \left(\int_0^\infty e^{-\xi/2} \operatorname{Re} c_1(\xi) e^{-\xi/2}\,d\xi\right)^2 \\
&\qquad [\text{ using Schwarz's inequality }] \\
&\leq 1 - \int_0^\infty e^{-2\xi}(\operatorname{Re} c_1(\xi))^2\,d\xi + \int_0^\infty e^{-\xi}(\operatorname{Re} c_1(\xi))^2\,d\xi \int_0^\infty e^{-\xi}\,d\xi \\
&\leq 1 - \int_0^\infty e^{-2\xi}(\operatorname{Re} c_1(\xi))^2\,d\xi + \int_0^\infty e^{-\xi}(\operatorname{Re} c_1(\xi))^2\,d\xi
\end{aligned}
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\begin{aligned}
a_3 &= \operatorname{Re} a_3 = -\int_0^\infty e^{-2\xi} \operatorname{Re} c_2(\xi) d\xi + \operatorname{Re} \left(\int_0^\infty e^{-\xi} c_1(\xi) d\xi \right)^2 \\
&\qquad \left[\text{ using } (\operatorname{Re} c_1)^2 \le 2 + \operatorname{Re} c_2 \text{ and } \operatorname{Re}(w^2) \le (\operatorname{Re} w)^2 \right] \\
&\le \int_0^\infty e^{-2\xi} (2 - (\operatorname{Re} c_1(\xi))^2) d\xi + \left(\int_0^\infty e^{-\xi} \operatorname{Re} c_1(\xi) d\xi \right)^2 \\
&= 1 - \int_0^\infty e^{-2\xi} (\operatorname{Re} c_1(\xi))^2 d\xi + \left(\int_0^\infty e^{-\xi/2} \operatorname{Re} c_1(\xi) e^{-\xi/2} d\xi \right)^2 \\
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&\le 1 - \int_0^\infty e^{-2\xi} (\operatorname{Re} c_1(\xi))^2 d\xi + \int_0^\infty e^{-\xi} (\operatorname{Re} c_1(\xi))^2 d\xi \\
&= 1 + \int_0^\infty (e^{-\xi} - e^{-2\xi}) (\operatorname{Re} c_1(\xi))^2 d\xi \le 1 + 4 \int_0^\infty (e^{-\xi} - e^{-2\xi}) d\xi = 3.
\end{aligned}
$$

From Loewner PDE to Loewner chains

In the section we prove that any Loewner PDE has a solution which is a Loewner chain.

We have to solve firstly a certain ODE.

From Loewner PDE to Loewner chains

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We apply this result to get univalence criteria for analytic maps.

From Loewner PDE to Loewner chains

Theorem

Let $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ *be a Herglotz function. Then, for any* $z \in \mathbb{D}$ *and any* $s \geq 0$ *the initial value problem*

$$\frac{dw}{dt} = -wp(w, t) \quad \text{for a.e. } t \in [s, +\infty), \qquad w(s) = z \qquad \text{(Loewner ODE)}$$

has a unique solution $w(t)$ *in* $t \in [s, +\infty)$.

From Loewner PDE to Loewner chains

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(Loewner ODE)

has a unique solution $w(t)$ in $t \in [s, +\infty)$. Write $\varphi_{s,t}(z) := w(t)$. Then $\varphi_{s,t}$ is a univalent for all $0 \leq s \leq t < +\infty$ and

EF1 $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$,

EF2 $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ *whenever* $0 \leq s \leq u \leq t < +\infty$,

EF3 $\varphi_{s,t}(0) = 0$ *and* $\varphi'_{s,t}(0) = e^{s-t}$.

Definition

A family of analytic self-maps of the unit disk $(\varphi_{s,t})_{0 \leq s \leq t < +\infty}$ satisfying EF1, EF2, and EF3 is called an evolution family.

From Loewner PDE to Loewner chains

Proof.
We define $w_0(z,t) \equiv 0$ and, recursively,

$$w_{n+1}(z,t) = z \exp\left[-\int_s^t p(w_n(z,\tau),\tau)d\tau\right] \quad n = 0,1,2,3...$$

for $t \in [s,+\infty)$ and for all $z \in \mathbb{D}$. Notice that $|w_n(z,t)| < 1$ for all n and the function $z \mapsto w_n(z,t)$ is analytic for all t.

From Loewner PDE to Loewner chains

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We define $w_0(z,t) \equiv 0$ and, recursively,

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for $t \in [s, +\infty)$ and for all $z \in \mathbb{D}$. Notice that $|w_n(z,t)| < 1$ for all n and the function $z \mapsto w_n(z,t)$ is analytic for all t.

Schwarz's lemma implies that $|w_n(z,t)| < |z|$.

From Loewner PDE to Loewner chains

Proof. Using again the representation formula $p(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)$ we deduce that

$$|p'(z,\tau)| \leq 2(1-|z|)^{-2}.$$

From Loewner PDE to Loewner chains

Proof. Using again the representation formula $p(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)$ we deduce that

$$|p'(z,\tau)| \leq 2(1-|z|)^{-2}.$$

Moreover $|e^{-a} - e^{-b}| \leq |a-b|$ whenever $\operatorname{Re} a, \operatorname{Re} b \geq 0$. Thus

$$|w_{n+1}(z,t) - w_n(z,t)| =$$

$$= |z| \left| \exp\left[-\int_s^t p(w_n(z,\tau),\tau)d\tau\right] - \exp\left[-\int_s^t p(w_{n-1}(z,\tau),\tau)d\tau\right]\right|$$

$$\leq \int_s^t |p(w_n(z,\tau),\tau) - p(w_{n-1}(z,\tau),\tau)|\, d\tau$$

$$= \int_s^t \left| \int_{w_{n-1}(z,\tau)}^{w_n(z,\tau)} p'(\xi,\tau)d\xi \right| d\tau \leq \int_s^t \int_{w_{n-1}(z,\tau)}^{w_n(z,\tau)} \left| \frac{2}{(1-|\xi|)^2} \right| d\xi d\tau$$

$$\leq \int_s^t \int_{w_{n-1}(z,\tau)}^{w_n(z,\tau)} \left| \frac{2}{(1-|z|)^2} \right| d\xi d\tau \leq \frac{2}{(1-|z|)^2} \int_s^t |w_n(z,\tau) - w_{n-1}(z,\tau)|\, d\tau.$$

From Loewner PDE to Loewner chains

Proof.

$$|w_{n+1}(z,t) - w_n(z,t)| \leq \frac{2}{(1-|z|)^2} \int_s^t |w_n(z,\tau) - w_{n-1}(z,\tau)|\, d\tau.$$

By induction we have that

$$|w_{n+1}(z,t) - w_n(z,t)| \leq \frac{2^n (t-s)^n}{(1-|z|)^{2n} n!} \quad (z \in \mathbb{D},\ n = 0, 1, 2, 3, \ldots).$$

We conclude that $\lim_n w_n(z,t)$ exists uniformly in $|z| \leq r$, $s \leq t \leq T$ for every $r < 1$ and T.

From Loewner PDE to Loewner chains

Proof.

$$|w_{n+1}(z,t) - w_n(z,t)| \leq \frac{2}{(1-|z|)^2} \int_s^t |w_n(z,\tau) - w_{n-1}(z,\tau)|\, d\tau.$$

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We conclude that $\lim_n w_n(z,t)$ exists uniformly in $|z| \leq r$, $s \leq t \leq T$ for every $r < 1$ and T.
Denoting the limit by $\varphi_{s,t}(z)$, we obtain an analytic function in z and by the Lebesgue's bounded convergence theorem we obtain that

$$\varphi_{s,t}(z) = z \exp\left[- \int_s^t p(\varphi_{s,\tau}(z),\tau) d\tau \right].$$

From this equation, it is easy to deduce the properties of $\varphi_{s,t}$.

From Loewner PDE to Loewner chains

Corollary

Take p and $(\varphi_{s,t})$ as in the above theorem. Then

$$f_s(z) = \lim_{t \to +\infty} e^t \varphi_{s,t}(z)$$

exists uniformly on compacta of $\mathbb{D}$, (f_t) is a Loewner chain satisfying $f_s = f_t \circ \varphi_{s,t}$ and

$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) p(z,t).$$

From Loewner PDE to Loewner chains

Proof.

We just obtain the existence of the limit $f_s(z) = \lim_{t\to+\infty} e^t \varphi_{s,t}(z)$ and $f_s = f_t \circ \varphi_{s,t}$. The rest of the proof is easy.

By the proof of the above theorem, we have that

$$\varphi_{s,t}(z) = z \exp\left[-\int_s^t p(\varphi_{s,\tau}(z),\tau)d\tau\right].$$

Therefore

$$e^{t-s}\varphi_{s,t}(z) = z \exp\left[\int_s^t (1 - p(\varphi_{s,\tau}(z),\tau))d\tau\right]$$

and this function belongs to $\mathcal{S}$. By Koebe distortion theorem, we have

$$|\varphi_{s,t}(z)| \le \frac{|z|}{(1-|z|)^2} e^{s-t}.$$

From Loewner PDE to Loewner chains

Proof.

Fix $r < 1$. If $|z| < r$ then $|\varphi_{s,t}(z)| \leq r$ and

$$|1-p(\varphi_{s,\tau}(z),\tau)| = \left|\int_0^{\varphi_{s,\tau}(z)} p'(\xi,\tau)d\xi\right| \leq \frac{2}{(1-r)^2}|\varphi_{s,\tau}(z)| \leq \frac{2}{(1-r)^4}e^{s-\tau}.$$

Thus, if $|z| \leq r$ and $t, t' \geq s$, after some computations, we have

$$|e^{t-s}\varphi_{s,t}(z) - e^{t'-s}\varphi_{s,t'}(z)| = ... =$$

$$= |e^{t-s}\varphi_{s,t}(z)|\left|1 - \exp\left[\int_t^{t'}(1-p(\varphi_{s,\tau}(z),\tau))d\tau\right]\right|$$

$$\leq \frac{|z|}{(1-|z|)^2}\left|1 - \exp\left[\int_t^{t'}(1-p(\varphi_{s,\tau}(z),\tau))d\tau\right]\right|.$$

Thus the family $(e^{t-s}\varphi_{s,t})_{t\geq s}$ is Cauchy in the compact set $r\overline{\mathbb{D}}$.

From Loewner PDE to Loewner chains

Proof.
Thus the limit

$$f_s(z) = \lim_{t \to +\infty} e^t \varphi_{s,t}(z)$$

is well defined. Moreover, if $s < t$,

$$f_s(z) = \lim_{\tau \to +\infty} e^\tau \varphi_{s,\tau}(z) = \lim_{\tau \to +\infty} e^\tau \varphi_{t,\tau} \circ \varphi_{s,t}(z) = f_t(\varphi_{s,t}(z)).$$

Univalence

So far, we have applied Loewner theory just in one direction: if (f_t) is a Loewner chain then it is the solution of a certain PDE

$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) p(z,t).$$

Moreover, we have learned that such PDE has a solution which is a Loewner chain.

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Moreover, we have learned that such PDE has a solution which is a Loewner chain.
But it can have many others solutions: take h an entire function and write $g_t := h \circ f_t$. Then

$$\frac{\partial g_t(z)}{\partial t} = h'(f_t(z))\frac{\partial f_t(z)}{\partial t} = h'(f_t(z)) z f_t'(z) p(z,t) = z g_t'(z) p(z,t).$$

The functions g_t are not univalent unless $h(z) = az + b$.

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The functions g_t are not univalent unless $h(z) = az + b$.

The goal of this section is to characterize Loewner chains between all the solution of the above PDE and, as a by product of such result, we plan to get univalence criteria.

Univalence

Theorem (Univalence Theorem)

Let $0 < r < 1$. Let (f_t) be a family of analytic maps in $r\mathbb{D}$ s.t.

1. $f_t(0) = 0$ *and* $f_t'(0) = e^t$;
2. *For each* $z \in r\mathbb{D}$, *the map* $t \mapsto f_t(z)$ *is abs. continuous;*
3. *There is* K *such that*

$$|f_t(z)| \leq Ke^t \quad \text{for all } |z| < r, \text{ and } t \geq 0;$$

4. *There exists a Herglotz function* $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ *s.t.*

$$\frac{\partial f_t(z)}{\partial t} = zf_t'(z)p(z,t) \quad \text{for all } |z| < r, \text{ and } t \geq 0.$$

Then the function f_t *has an analytic continuation to* $\mathbb{D}$ *and* (f_t) *is a Loewner chain.*

Univalence

Proof. Take $(\varphi_{s,t})$ the evolution family associated with p. Fix $z \in r\mathbb{D}$.
Using again Vitali's theorem, $t \mapsto k(t) := f_t(\varphi_{s,t}(z))$ is absolutely continuous in every interval of the form $[s, T]$.
Then

$$\begin{aligned} k'(t) &= f_t'(\varphi_{s,t}(z))\frac{\partial}{\partial t}(\varphi_{s,t}(z)) + \frac{\partial f_t}{\partial t}(\varphi_{s,t}(z)) \\ &= -f_t'(\varphi_{s,t}(z))\varphi_{s,t}(z)p(\varphi_{s,t}(z), t) \\ &\qquad +\varphi_{s,t}(z)f_t'(\varphi_{s,t}(z))p(\varphi_{s,t}(z), t) = 0. \end{aligned}$$

But $k(s) = f_s(z)$. Then $f_s(z) = k(s) = k(t) = f_t \circ \varphi_{s,t}(z)$ whenever $|z| < r$.

Univalence

Proof. By Schwarz's lemma (applied twice), if $h : \mathbb{D} \to \mathbb{D}$ is analytic and $h(0) = h'(0) = 0$, then $|h(w)| \leq |w|^2$ for all $w \in \mathbb{D}$.

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Write $h(w) = \frac{e^{-t} f_t(rw) - rw}{K+1}$. By hypothesis, we have h is an analytic self-map of $\mathbb{D}$ such that $h(0) = h'(0) = 0$. Thus, taking $z = rw$ we have

$$|e^{-t} f_t(z) - z| \leq (K+1) r^{-2} |z|^2.$$

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Hence, using this inequality and applying Koebe distortion theorem to the function $e^{s-t}\varphi_{s,t}$, we have

$$\begin{aligned} |f_s(z) - e^t \varphi_{s,t}(z)| &= e^t |e^{-t} f_t(\varphi_{s,t}(z)) - \varphi_{s,t}(z)| \\ &\leq e^t (K+1) r^{-2} |\varphi_{s,t}(z)|^2 \\ &\leq e^t (K+1) r^{-2} |z|^2 (1 - |z|)^{-4} e^{2s-2t} \\ &\leq (K+1)(1-r)^{-4} e^{2s-t}. \end{aligned}$$

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Therefore, $e^t \varphi_{s,t}(z) \to f_s(z)$ as $t \to \infty$ for $|z| < r$.

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Therefore, $e^t \varphi_{s,t}(z) \to f_s(z)$ as $t \to \infty$ for $|z| < r$.
This implies that f_s is nothing but the restriction to $r\mathbb{D}$ of the function built in the last corollary .

Starlike functions

A function f is called spirallike of type $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if it is univalent in $\mathbb{D}$ and $\exp(-e^{i\alpha}t)f(\mathbb{D}) \subseteq f(\mathbb{D})$ for all $t \geq 0$. Starlike functions are the special case $\alpha = 0$.

Corollary (Spacek, 1933, Robertson, 1961)

The function $f(z) = z + \ldots$ analytic in $\mathbb{D}$ is spirallike of type α if and only if

$$\operatorname{Re}\left[e^{i\alpha} z \frac{f'(z)}{f(z)}\right] > 0 \quad \text{for all } z \in \mathbb{D}.$$

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Proof. Define $f_t(z) = e^t f(z) = e^t z + \dots$ for all $z \in \mathbb{D}$ and $t \geq 0$. Since

$$\frac{\partial f_t(z)/\partial t}{z f_t'(z)} = \frac{f(z)}{z f'(z)},$$

defining $p(z,t) = \frac{f(z)}{zf'(z)}$ it is a non-negative function for which

$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) p(z,t).$$

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Clearly (f_t) satisfies the hypothesis of Univalence Theorem and f_t is a Loewner chain. This implies that $f = f_0$ is univalent and $f(\mathbb{D}) = f_0(\mathbb{D}) \subseteq f_t(\mathbb{D}) = e^t f(\mathbb{D})$, that is, $e^{-t} f(\mathbb{D}) \subseteq f(\mathbb{D})$ for all t.

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Univalence criteria

From distortion theorem, one can prove that any $f \in \mathcal{S}$ satisfies

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| < 6 \qquad (z \in \mathbb{D}).$$

Next result shows a partial converse.

Corollary (Becker, 1972)

Let f be analytic in $\mathbb{D}$ with $f'(0) \neq 0$. If

$$(1-|z|^2)\left|z\frac{f''(z)}{f'(z)}\right| < 1 \qquad (z \in \mathbb{D})$$

then f is univalent in $\mathbb{D}$.

Univalence criteria

Proof. We may assume that $f'(0) = 1$. Let

$$f_t(z) := f(e^{-t}z) + (e^t - e^{-t})zf'(e^{-t}z) \quad \text{for all } z \in \mathbb{D} \text{ and } t \geq 0.$$

Then

$$\frac{\partial f_t(z)}{\partial t} = e^t zf'(e^{-t}z) - (1 - e^{-2t})z^2 f''(e^{-t}z),$$

$$zf_t'(z) = e^t zf'(e^{-t}z) + (1 - e^{-2t})z^2 f''(e^{-t}z).$$

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Write $p(z,t) := \frac{\partial f_t(z)}{\partial t} / (zf_t'(z))$. Then

$$\left| \frac{p(z,t) - 1}{p(z,t) + 1} \right| = \left| \frac{\frac{\partial f_t(z)}{\partial t} - zf_t'(z)}{\frac{\partial f_t(z)}{\partial t} + zf_t'(z)} \right| =$$

$$= (1 - e^{-2t}) \left| e^{-t} z \frac{f''(e^{-t}z)}{f'(e^{-t}z)} \right| \leq \frac{1 - e^{-2t}}{1 - e^{-2t}|z|^2} < 1.$$

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The map $T(z) = \frac{z+1}{1-z}$ sends the unit disk into the right half-plane. But $p(z,t) = T\left(\frac{p(z,t)-1}{p(z,t)+1}\right)$.

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Univalence criteria

Proof. ...
This shows that (f_t) satisfies the hypothesis of Univalence Theorem.

Thus (f_t) is a Loewner chain and, in particular, $f = f_0$ is univalent.

Radial Loewner theory

Summing up, we have introduced three concepts: evolutions families, Herglotz functions and Loewner chains.
There is a one to one correspondence between these notions:

Loewner chains (f_t) $\xleftrightarrow{\varphi_{s,t}=f_t^{-1}\circ f_s}$ Evolution families $(\varphi_{s,t})$

Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = zp(z,t)f_t'(z)$$

Loewner ODE

$$\frac{dw}{dt} = -zp(w,t), \; w(s) = z$$

$$\varphi_{s,t}(z) := w(t)$$

Herglotz function $p(w,t)$

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Loewner's theory has been used to prove several deep results in various branches of mathematics, even apparently unrelated to complex analysis. So far, we have presented two of them but there are many others. See Duren and Pommerenke.